Article | Received 6 June 2025; Revised 27 September 2025; Accepted 29 September 2025; Published 16 October 2025
https://doi.org/10.55092/aias20250008

A flatness-based adaptive fuzzy control method for an endogenous economic growth model

Gerasimos Rigatos[1,*], **Farouk Zouari**[2] and **Pierluigi Siano**[3]

[1] Unit of Industrial Automation, Industrial Systems Institute, Rion, Patras 26504, Greece

[2] Laboratory of Automatic Control, University of Tunis El Manar, Tunis 1068, Tunisia

[3] Department of Management, University of Salerno, Fisciano 84084, Italy

* Correspondence author; E-mail: grigat@ieee.org; grigat@athenarc.gr.

Highlights:

- The paper develops a flatness-based adaptive fuzzy control method for the Uzawa–Lucas endogenous growth model.
- It proves that the Uzawa–Lucas model is differentially flat and derives its equivalent input–output linearized representation.
- An output-feedback indirect adaptive fuzzy controller is designed, achieving estimation of unknown dynamics and ensuring global asymptotic stability through Lyapunov analysis.

Abstract: The article proposes a flatness-based adaptive control approach for the Uzawa-Lucas financial growth model. This model describes interaction between physical and human capital within an economy and so far equilibria and bifurcations have been computed about it. With the article's developments the problem of control and stabilization of the Uzawa-Lucas growth dynamics is solved through the application of exogenous control inputs and after considering uncertainty and unknown parameters for the related state-space model. It is proven that the Uzawa-Lucas state-space model is differentially flat and its transformation into an equivalent input-output linearized form is achieved. For the latter description of this financial system an indirect adaptive fuzzy control method is developed. The method relies on feedback of only the system's output and accomplishes simultaneously the estimation of unknown dynamics of the growth model as well as the convergence of the state variables to the defined setpoints. Learning of the unknown dynamics is performed through neurofuzzy approximators which are iteratively updated with the use of gradient algorithms. Since only feedback of the system's output is used, state vector estimation was performed with the use of a convergent state observer. The computation of the gains of the adaptive fuzzy controller required the solution of two algebraic Riccati equations. The global asymptotic stability properties of the control method are proven through Lyapunov analysis.

keywords: economy's endogenous growth model; Uzawa-Lucas model; differential flatness properties; flatness-based adaptive fuzzy control; output feedback; state-observer; global stability

1. Introduction

Endogenous growth as the result of the interaction between physical and human capital has been a long-standing subject of research [1–4]. The Uzawa-Lucas model is among the most acknowledged approaches for representing the dynamics of endogenous growth [5,6]. The main concept of this model is that individuals within an economy contribute either to the development of physical capital or to the development of knowhow and skills. The interaction between the two activities results to the accumulation of physical capital (that is consumption goods) and to the accumulation of human capital (that is skillful people capable of contributing to further economic growth) [7,8]. Extended forms of the Uzawa-Lucas model present interaction between physical capital, human capital, per capita consumption, allocation of individual's time to either physical or human capital production and finally labor [9–11]. There exist also concise forms of these models showing the evolution in time and the cyclical behavior of variables related with the physical capital, the human capital, and the distribution of the individuals' time to either physical or human capital production [12,13]. The stability properties of the Uzawa-Lucas model have also extensively studied and related results have been obtained about the existence of equilibria and the associated bifurcations [14,15]. Parameters related with the tendency of the individuals for creating savings or for consuming determine the model's stability properties. Actually, the prioritization for creating savings contributes to the human capital development instead of raising consumption and does not favor the growth of physical production. As a result of structural (parametric) changes the model's state variables keep on changing in time until an equilibrium is reached.

Due to the complex nonlinear structure of the Uzawa-Lucas model there exist hardly any results on its control and stabilization with the use of external control inputs. The present article comes to propose a solution to the nonlinear adaptive control problem of the Uzawa-Lucas endogenous growth model after using as control inputs parameters related with the propensity for creating savings or with the propensity for consumption. Considering next, that the parameters of the Uzawa-Lucas growth model are unknown or that there is uncertainty due to unmodelled dynamics, the solution of the related control problem becomes of high difficulty. To handle this model imprecision together with stabilization of the growth system's dynamics one can opt for adaptive control techniques [16–18]. So far there have been some results on adaptive neural or adaptive fuzzy control of financial systems [19–23]. A significant part of such approaches is addressed to chaotic finance dynamics while little has been done on adaptive control of growth models [24–26].

The present article develops an adaptive fuzzy control method for the Uzawa-Lucas endogenous growth model. The method is based on differential flatness theory and on diffeomorphisms (change of state variables) which allow for transformation of the initial nonlinear description of the system, into an equivalent linear form. Moreover, the method is implemented only with output feedback thus requiring to monitor only a limited number of state variables in the financial system. First, it is proven that the dynamic model of the Uzawa-Lucas endogenous growth model is a differentially flat one. This means that all its state variables and its control inputs can be expressed as differential functions of a primary state variable which is the system's flat output. Moreover, the flat output and its derivatives are differentially independent which means that they are not connected between them with a relation of the type of an homogeneous differential equation [27,28]. Next, by applying a change of state variables (diffeomorphism) which is in accordance to differential flatness theory, one arrives at an input-output linearized system. This description is also written in the linear canonical (Brunovsky) state-space form [29–31]. For the latter description the design of a stabilizing state feedback controller becomes possible.

Since there is no knowledge about the financial system's dynamics and the control method is a model-free one, the unknown parts of the dynamics are identified in real-time with the use of neurofuzzy approximators. The information obtained about the system's dynamics is used for the computation of the control input, and thus an indirect adaptive control scheme is established. The update of the approximators' weights is based on a gradient-type algorithm [32–34]. The learning rate of the neurofuzzy approximators is obtained from the requirement the first derivative of the system's Lyapunov function to be always negative. The computation of the control signal requires also the solution of two algebraic Riccati equations. Lyapunov stability analysis proves that the control loop satisfies the H-infinity tracking performance ciretion and this signifies robustness against model uncertainty and external perturbations. Moreover, under moderate conditions global asymptotic stability is proven.

The structure of the paper is as follows: In Section 2, the dynamics of the Uzawa-Lucas endogenous growth model is analyzed and the related state-space model is formulated. In Section 3, the differential flatness properties of the Uzawa-Lucas model are proven and the transformation of the system's state-space model into an input-output linearized form is is completed. In Section 4, an output feedback-based adaptive fuzzy control approach is developed for the Uzawa-Lucas model. In Section 5, the global stability properties of the adaptive fuzzy control scheme are proven through Lyapunov analysis. In Section 6, the performance of the control method is further confirmed through simulation experiments. Finally, in Section 7, concluding remarks are stated.

2. The Uzawa-Lucas model of economic growth

2.1. The generic Uzawa-Lucas model

The Uzawa-Lucas endogenous growth model is a two-sector model and comprises (i) the physical capital sector which produces physical capital (ii) the human capital sector which produces human capital. In this model, individuals are considered to have the same level of work qualifications and expertise. Some of their time is used for producing goods (physical capital) while some other part of their time is used for training and developing skills (human capital). One can usually find two interpretations of the Uzawa-Lucas model (a) the social planner's problem, and (b) the representative agent's problem [1,2,14].

The production function in the production sector is defined as

$$Y = AK^{a}(\varepsilon{\cdot}h{\cdot}L)^{(1-a)}h_a^{\zeta} \tag{1}$$

with $0 < a < 1$, where Y is the output (physical product), A is the technology level, K is the physical capital, a is the share of physical capital, L is the labor, h is the human capital per person, ε is the function of time devoted for developing physical products, $1-\varepsilon$ is the function of time devoted for training and skill development purposes, variable C denotes consumption, $\varepsilon \cdot h \cdot L$ is a quantity of labor measured in efficiency units, h_a^{ζ} measures the externality associated with average human capital of the workforce h_a, ζ is the positive externality parameter in the production of human capital (It is related to the tendency for saving money. Actually, the more one saves, the more he reduces consumption and invests on human capital).

The equation about physical capital accumulation is [1,2,14]

$$\dot{K} = AK^a(\varepsilon \cdot h \cdot L)^{(1-a)} h_n^{\zeta} - C - \delta K \tag{2}$$

In per capita (normalized form) the previous equation becomes [1,2,14]

$$\dot{k} = Ak^a(\varepsilon \cdot h)^{(1-a)} h_n^{\zeta} - c - (n+\delta)k \tag{3}$$

The human capital accumulation equation is given by [1,2,14]

$$\dot{h} = \eta h(1-\varepsilon) \tag{4}$$

where η is defined as the schooling (knowledge transmission productivity). The decision problem is defined as

$$max_{a_t,\varepsilon_t} \int_t^{\infty} \frac{e^{-(\rho-n)t}(c(\tau)^{(1-\sigma)}-1)}{1-\sigma} \tag{5}$$

subject to the dynamics of the physical and human capital variation, that is

$$\dot{k} = Ak^a(\varepsilon \cdot h)^{(1-a)} h_n^{\zeta} - c - (n+\delta)k \tag{6}$$

$$\dot{h} = \eta h(1-\varepsilon) \tag{7}$$

ρ $(\rho > n > 0)$ is the subjective discount rate, $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption.

2.2. The social planner's problem

In this approach, the social planner takes into account the externality associated with human capital when solving the maximization problem of Equation (5) subject to Equation (6) and Equation (7). The equations that describe the Uzawa-Lucas model from a social planner's approach are as follows [1,2,14]:

$$\frac{\dot{k}}{k} = Ak^{a-1}\varepsilon^{1-\alpha}h^{(1-a-\zeta)} - \frac{c}{k} - (n+\delta) \tag{8}$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon) \tag{9}$$

$$\frac{\dot{c}}{c} = \frac{aAk^{(a-1)}\varepsilon^{(1-a)}h^{1-a-\zeta} - (\rho+\delta)}{\sigma} \tag{10}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(1-a+\zeta)}{(1-a)}\varepsilon + \eta\frac{(1-a+\zeta)}{a} - \frac{c}{k} + \frac{(1-a)}{a}(n+\delta) \tag{11}$$

$$\frac{\dot{L}}{L} = n \tag{12}$$

Next, the following variables are defined:

$$m = \frac{Y}{K} \quad g = \frac{c}{k} \tag{13}$$

Taking logarithms of m and g and differentiating with respect to time, the dynamics of the Uzawa-Lucas model becomes [1,2,14]:

$$\frac{\dot{m}}{m} = -(1-a)m + \frac{1-a}{a}(n+\delta) + n\frac{(1-a+\zeta)}{a} \tag{14}$$

$$\frac{\dot{g}}{g} = (\frac{a}{\sigma} - 1)n - \frac{\rho}{\sigma} - \delta(\frac{1}{\sigma} - 1) + g + n \tag{15}$$

From the equilibrium condition $\dot{m} = 0$ and $\dot{g} = 0$ one obtains the fixed point [14]

$$m^* = \eta\frac{(1-a+\zeta)}{a} + \frac{(n+\delta)}{a} \tag{16}$$

$$g^* = \frac{\rho-\eta}{\sigma} + \frac{1-a}{a}(n+\delta) + n\frac{(1-a+\zeta)}{a(1-a)}\frac{(\sigma-a)}{\sigma} \tag{17}$$

The stability properties of the equilibrium can be found by analyzing the eigenvalues of the Jacobian matrix of the system

$$\begin{gathered} \dot{m} = [-(1-a)m + \frac{(1-\alpha)}{a}(n+\delta) + n\frac{(1-a+\zeta)}{a}]m \\ \dot{g}[(\frac{a}{\sigma} - 1)m - \frac{\rho}{\sigma} - \delta(\frac{1}{\sigma} - 1) + g + n]g \end{gathered} \tag{18}$$

that is

$$J = \begin{pmatrix} \frac{\partial \dot{m}}{\partial m} & \frac{\partial \dot{m}}{\partial g} \\ \\ \frac{\partial \dot{g}}{\partial m} & \frac{\partial \dot{g}}{\partial g} \end{pmatrix} |_{(m^*,g^*)} \tag{19}$$

2.3. The representative agent's problem

The economy of the Uzawa-Lucas model from a decentralized market's perspective can be defined as follows [1,2,14]:

$$\frac{\dot{k}}{k} = Ak^{a-1}\varepsilon^{1-\alpha}h^{(1-a-\zeta)} - \frac{c}{k} - (n+\delta) \tag{20}$$

$$\frac{\dot{h}}{h} = \eta(1-\varepsilon) \tag{21}$$

$$\frac{\dot{c}}{c} = \frac{aAk^{(a-1)}\varepsilon^{(1-a)}h^{1-a-\zeta} - (\rho+\delta)}{\sigma} \tag{22}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(a-\zeta)}{(1-a)}\varepsilon + \eta\frac{(1-a+\zeta)}{a} - \frac{c}{k} + \frac{(1-a)}{a}(n+\delta) \tag{23}$$

$$\frac{\dot{L}}{L} = n \tag{24}$$

Taking again logarithms of m and g and differentiating with respect to time, the following three equations define the dynamics of the Uzawa-Lucas model [1,2,14]:

$$\frac{\dot{m}}{m} = -(1-a)m + \frac{1-a}{a}(n+\delta) + n\frac{(1-a+\zeta)}{a} - \eta\frac{\zeta}{a}\varepsilon \tag{25}$$

$$\frac{\dot{g}}{g} = (\frac{a}{\sigma} - 1)m - \frac{\rho}{\sigma} - \delta(\frac{1}{\sigma} - 1) + g + n \tag{26}$$

$$\frac{\dot{\varepsilon}}{\varepsilon} = \eta\frac{(a-\zeta)}{a}\varepsilon + \eta\frac{(1-a+\zeta)}{a} - g + \frac{(1-a)}{a}(n+\delta) \tag{27}$$

where parameter ρ denotes the discount rate. Using the equilibrium condition for this model, that is $\dot{m}=0$, $\dot{g}=0$ and $\dot{\varepsilon}=0$ one obtains the following fixed point

$$\varepsilon^* = 1 - \frac{(1-a)(\rho-n-\eta)}{\eta[\zeta-\sigma(1-a+\zeta)]} \tag{28}$$

$$m^* = \eta\frac{[1-a+\zeta(1-\varepsilon^*)]}{a(1-a)} + \frac{n}{a} \tag{29}$$

$$g^* = \eta\frac{[1-a+\zeta(1-\varepsilon^*)+a\varepsilon^*]}{a(a-1)} + \frac{n(1-a)}{a} \tag{30}$$

Next, using the state-space description [14]:

$$\dot{m} = [-(1-a)m + \frac{1-a}{a}(n+\delta) + \tilde{\eta}\frac{(1-a+\zeta)}{a} - \tilde{\eta}\frac{\zeta}{a}\varepsilon]m \tag{31}$$

$$\dot{g} = [(\frac{a}{\sigma}-1)m - \frac{\rho}{\sigma} - \delta(\frac{1}{\sigma}-1) + g + n]g \tag{32}$$

$$\dot{\varepsilon} = [\tilde{\eta}\frac{(a-\zeta)}{a}\varepsilon + \tilde{\eta}\frac{(1-a+\zeta)}{a} - g + \frac{(1-a)}{a}(n+\delta)]\varepsilon \tag{33}$$

one can compute the related Jacobian matrix at the equilibrium, that is

$$J = \begin{pmatrix} \frac{\partial \dot{m}}{\partial m} & \frac{\partial \dot{m}}{\partial g} & \frac{\partial \dot{m}}{\partial \varepsilon} \\ \\ \frac{\partial \dot{g}}{\partial m} & \frac{\partial \dot{g}}{\partial g} & \frac{\partial \dot{g}}{\partial \varepsilon} \\ \\ \frac{\partial \dot{\varepsilon}}{\partial m} & \frac{\partial \dot{\varepsilon}}{\partial g} & \frac{\partial \dot{\varepsilon}}{\partial \varepsilon} \end{pmatrix} |_{(m^*,g^*),\varepsilon^*} \tag{34}$$

and from the associated eigenvalues can conclude the stability properties of the system around the equilibrium.

2.4. *State-space description of the Uzawa-Lucas model in matrix form*

In the Uzawa-Lucas model of Equation (31) to Equation (33), the following state variables are defined: $x_1 = m$, $x_2 = g$ and $x_3 = \varepsilon$, while the control input is taken to be the discount rate parameter $u = \rho$ that appears in the second row of the state-space model. Thus, the state-space description of the Uzawa-Lucas model becomes

$$\dot{x}_1 = [-(1-a)x_1 + \frac{(1-a)}{a}(n\delta) + \tilde{\eta}\frac{(1-a+\zeta)}{a} - \tilde{\eta}x_3]x_1 \tag{35}$$

$$\dot{x}_2 = [(\frac{a}{\sigma}-1)x_1 - \frac{u}{\sigma} - \delta(\frac{1}{\sigma}-1) + x_2 + n]x_2 \tag{36}$$

$$\dot{x}_3 = [\tilde{\eta}\frac{(a-\zeta)}{a}x_3 + \tilde{\eta}\frac{(1-a+\zeta)}{a} - x_2 + \frac{(1-a)}{\sigma}(n+\tilde{\delta})]x_3 \tag{37}$$

After, reordering of terms one obtains the following state-space description in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -(1-a)x_1^2 + \frac{(1-a)}{a}(n\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3 \\ (\frac{a}{\sigma}-1)x_1x_2 - \delta(\frac{1}{\sigma}-1)x_2 + x_2^2 + nx_2 \\ \tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(n+\tilde{\delta})x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{x_2}{\sigma} \\ 0 \end{pmatrix} u \tag{38}$$

or equivalently, the state-space description of the Uzawa-Lucas model can be written in the concise form:

$$\dot{x} = \tilde{f}(x) + \tilde{g}(x)u \tag{39}$$

with $x \in R^{3\times 1}$, $\tilde{f}(x) \in R^{3\times 1}$, $\tilde{g}(x) \in R^{3\times 1}$ and $u \in R$.

3. Differential flatness properties of the Uzawa-Lucas state-space model

3.1. Proof of the differential flatness properties of the Uzawa-Lucas model

The previously defined state-space description of the Uzawa-Lucas endogenous growth model is differentially flat, with flat output $y = x_1$. Indeed from Equation (35) one has

$$\begin{array}{c} \frac{\tilde{\eta\zeta}}{a} x_1 x_3 = -\dot{x}_1 - (1-a)x_1^2 + \frac{(1-a)}{a}(n+\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 \Rightarrow \\ x_3 = \frac{-\dot{x}_1 - (1-a)x_1^2 + \frac{(1-a)}{a}(n+\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1}{\tilde{\eta}\frac{\zeta}{a}x_1} \Rightarrow \\ x_3 = h_3(x_1, \dot{x}_1) \end{array} \tag{40}$$

This signifies that state variable x_3 is a differential function of the flat output $y = x_1$. Moreover, from Equation (37) one has

$$\begin{array}{c} x_2 x_3 = \dot{x}_3 + \tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 + \frac{(1-a)}{a}(n+\delta)x_3 \Rightarrow \\ x_2 = \frac{\dot{x}_3 + \tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 + \frac{(1-a)}{a}(n+\delta)x_3}{x_3} \Rightarrow \\ x_2 = h_2(x_1, \dot{x}_1, \ddot{x}_1) \end{array} \tag{41}$$

Consequently, state variable x_2 is also a differential function of the system's flat output $y = x_1$. Moreover, from Equation (36) one has

$$\begin{array}{c} \frac{x_2}{\sigma}u = -dotx_2 + (\frac{a}{\sigma} - 1)x_1 x_2 - \delta(\frac{1}{\sigma} - 1)x_2 + x_2^2 + nx_2 \Rightarrow \\ u = \frac{-dotx_2 + (\frac{a}{\sigma} - 1)x_1 x_2 - \delta(\frac{1}{\sigma} - 1)x_2 + x_2^2 + nx_2}{\frac{x_2}{\sigma}} \Rightarrow \\ u = h_u(x_1, \dot{x}_1, \ddot{x}_1, x_1^{(3)}) \end{array} \tag{42}$$

Consequently, control input u is also a differential function of the system's flat output $y = x_1$. As a result of the above, all state-variables and the control input of the Uzawa-Lucas endogenous growth model can be written as functions of the flat output $y = x_1$ and of its derivatives. Besides, the flat output and its derivatives are differentially independent which means that they are not connected to each other through a relation in the form of a linear homogeneous differential equation. These two properties confirm that the Uzawa-Lucas model is differentially flat. The differential flatness of the model can be used for determining setpoints for its stabilizing feedback controller.

3.2. Input-output linearization of the Uzawa-Lucas model

The state-space equations that constitute the Uzawa-Lucas endogenous growth model are

$$\dot{x}_1 = -(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1 x_3 \tag{43}$$

$$\dot{x}_2 = (\frac{\alpha}{\sigma} - 1)x_1 x_2 - \delta(\frac{1}{\sigma} - 1)x_2 + x_2^2 + \eta x_2 - \frac{x_2}{\sigma}u \tag{44}$$

$$\dot{x}_3 = \tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})x_3 \tag{45}$$

By deriving in time Equation (43) one has

$$\ddot{x}_1 = -(1-a)2x_1\dot{x}_1 + \frac{(1-a)}{a}(\eta\delta)\dot{x}_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}\dot{x}_1 - \tilde{\eta}\dot{x}_1x_3 - \tilde{\eta}x_1\dot{x}_3 \tag{46}$$

By substituting the derivatives of the state variables in the above equations one obtains:

$$\begin{gathered}\ddot{x}_1 = (1-a)2x_1[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] + \\ \frac{(1-a)}{a}(\eta\delta)[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] + \\ \tilde{\eta}\frac{(1-a+\zeta)}{a}[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] - \\ \tilde{\eta}x_3[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] - \\ \tilde{\eta}x_1[\tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})x_3]\end{gathered} \tag{47}$$

By grouping terms one has

$$\begin{gathered}\ddot{x}_1 = [-(1-a)2x_1 + \frac{(1-a)}{a}(\eta\delta) + \frac{\tilde{\eta}(1-a+\zeta)}{a} - \tilde{\eta}x_3]\cdot \\ \cdot[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] - \\ -\eta x_1[\tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})x_3] - \\ -\tilde{\eta}x_1x_2x_3\end{gathered} \tag{48}$$

By deriving once again with respect to time one obtains:

$$\begin{gathered}x_1^{(3)} = [-(1-a)2\dot{x}_1 + \frac{(1-a)}{a}(\eta\delta) - \tilde{\eta}\dot{x}_3]\cdot \\ \cdot[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] + \\ [-(1-a)2x_1 + \frac{(1-a)}{a}(\eta\delta) + \frac{\tilde{\eta}(1-a+\zeta)}{a} - \tilde{\eta}x_3]\cdot \\ \cdot[-(1-a)2x_1\dot{x}_1 + \frac{(1-a)}{a}(\eta\delta)\dot{x}_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}\dot{x}_1 - \tilde{\eta}\dot{x}_1x_3 - \tilde{\eta}x_1\dot{x}_3] - \\ -\tilde{\eta}\dot{x}_1[\tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})x_3] - \\ \eta x_1[\tilde{\eta}\frac{(a-\zeta)}{a}2x_3\dot{x}_3 + \tilde{\eta}\frac{(1-a+\zeta)}{a}\dot{x}_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})\dot{x}_3] - \\ -\tilde{\eta}[\dot{x}_1x_2x_3 + x_1x_2\dot{x}_3] - \tilde{\eta}x_1x_3[(\frac{a}{\sigma}-1)x_1x_2 - \delta(\frac{1}{\sigma}-1)x_2 + x_2^2 + \eta x_2] - \\ -\tilde{\eta}x_1x_3\frac{x_2}{\sigma}u\end{gathered} \tag{49}$$

Thus, one arrives at the following input-output linearized form of the Uzawa-Lucas growth model

$$x_1^{(3)} = f(x) + g(x)u \tag{50}$$

where function $f(x)$ is given by

$$\begin{gathered}f(x) = [-(1-a)2\dot{x}_1 + \frac{(1-a)}{a}(\eta\delta) - \tilde{\eta}\dot{x}_3]\cdot \\ \cdot[-(1-a)x_1^2 + \frac{(1-a)}{a}(\eta\delta)x_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_1 - \tilde{\eta}x_1x_3] + \\ [-(1-a)2x_1 + \frac{(1-a)}{a}(\eta\delta) + \frac{\tilde{\eta}(1-a+\zeta)}{a} - \tilde{\eta}x_3]\cdot \\ \cdot[-(1-a)2x_1\dot{x}_1 + \frac{(1-a)}{a}(\eta\delta)\dot{x}_1 + \tilde{\eta}\frac{(1-a+\zeta)}{a}\dot{x}_1 - \tilde{\eta}\dot{x}_1x_3 - \tilde{\eta}x_1\dot{x}_3] - \\ -\tilde{\eta}\dot{x}_1[\tilde{\eta}\frac{(a-\zeta)}{a}x_3^2 + \tilde{\eta}\frac{(1-a+\zeta)}{a}x_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})x_3] - \\ \eta x_1[\tilde{\eta}\frac{(a-\zeta)}{a}2x_3\dot{x}_3 + \tilde{\eta}\frac{(1-a+\zeta)}{a}\dot{x}_3 - x_2x_3 + \frac{(1-a)}{\sigma}(\eta+\tilde{\delta})\dot{x}_3] - \\ -\tilde{\eta}[\dot{x}_1x_2x_3 + x_1x_2\dot{x}_3] - \tilde{\eta}x_1x_3[(\frac{a}{\sigma}-1)x_1x_2 - \delta(\frac{1}{\sigma}-1)x_2 + x_2^2 + \eta x_2]\end{gathered} \tag{51}$$

and function $g(x)$ is given by

$$g(x) = -\tilde{\eta} x_1 x_3 \tfrac{x_2}{\sigma} \tag{52}$$

3.3. Design of a stabilizing feedback controller

It has been shown that the Uzawa-Lucas endogenous growth model can be written in the input-output linearized form

$$x_1^{(3)} = f(x) + g(x)u \tag{53}$$

or, using the flat output definition $y = x_1$ and the differential flatness property of the model the following state-space description is obtained

$$y^{(3)} = f(y, \dot{y}, \ddot{y}) + g(y, \dot{y}, \ddot{y})u \tag{54}$$

where $f(y, \dot{y}, \ddot{y})$ and $g(y, \dot{y}, \ddot{y})$ where defined in Equation (51) and in Equation (52), respectively. By defining the transformed control input $v = \tilde{f}(y, \dot{y}, \ddot{y}) + \tilde{g}(y, \dot{y}, \ddot{y})u$ one has that

$$y^{(3)} = v \tag{55}$$

For the linearized description of the Uzawa-Lucas endogenous growth model given in Equation (55), and using the notation $z_1 = y$, $z_2 = \dot{y}$ and $z_3 = \ddot{y}$, and $v = f(y, \dot{y}, \ddot{y}) + g(y, \dot{y}, \ddot{y})u$ one arrives also at the state-space description (canonical Brunovsky form)

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v \tag{56}$$

$$z^{meas} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{57}$$

and the stabilizing feedback control input is given by

$$v = y_d^{(3)} - k_1(\ddot{y} - \ddot{y}_d) - k_2(\dot{y} - \dot{y}_d) - k_3(y - y_d) \tag{58}$$

and the control input that is actually applied to the financial system is

$$u = g^{-1}(y, \dot{y}, \ddot{y})[v - f(y, \dot{y}, \ddot{y})] \tag{59}$$

The previous control signal results in the tracking error dynamics of the form

$$e^{(3)}(t) + k_1\ddot{e}(t) + k_2\dot{e}(t) + k_3 e(t) = 0 \tag{60}$$

By selecting the feedback gains $k_i,\ i = 1, 2, 3$ such that the characteristic polynomial of Equation (60) to be a Hurwitz one, it is assured that $lim_{t\to\infty} e(t) = 0$.

4. Adaptive fuzzy control of the Uzawa-Lucas model using output feedback

4.1. Problem statement

Adaptive fuzzy control aims at solving the control problem of the Uzawa-Lucas endogenous growth model in case that its dynamics is unknown and the state vector is not completely measurable. It has been shown that after applying the differential flatness theory-based transformation, the following non-linear SISO system is obtained:

$$x^{(n)} = f(x,t) + g(x,t)u + \tilde{d} \tag{61}$$

where $f(x,t)$, $g(x,t)$ are unknown nonlinear functions and $\tilde{d}$ is an unknown additive disturbance. The objective is to make the system's output $y = x$ follow a given bounded reference signal x_d. In the presence of non-gaussian disturbances w, successful tracking of the reference signal is denoted by the H_∞ criterion [28]

$$\int_0^T e^T Qedt \leq \rho^2 \int_0^T w^T wdt \tag{62}$$

where ρ is the attenuation level and corresponds to the maximum singular value of the transfer function $G(s)$ of the linearized equivalent of Equation (61).

4.2. Transformation of tracking into a regulation problem

The H_∞ approach to nonlinear systems control consists of the following steps : (i) linearization is applied: (ii) the unknown system dynamics are approximated by neural of fuzzy estimators, (iii) an H_∞ control term, is employed to compensate for estimation errors and external disturbances. If the state vector is not measurable, this can be reconstructed with the use of an observer.

For measurable state vector x, desirable state vector x_m and uncertain functions $f(x,t)$ and $g(x,t)$ an appropriate control law for Equation (61) would be

$$u = \frac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) + K^T e + u_c] \tag{63}$$

where, $\hat{f}$ and $\hat{g}$ are the approximations of the unknown parts of the system dynamics f and g respectively, and which can be given by the outputs of suitably trained neuro-fuzzy networks. The term u_c denotes a supervisory controller which compensates for the approximation error $w = [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u$, as well as for the additive disturbance $\tilde{d}$. Moreover the vectors $K^T = [k_n, k_{n-1}, \cdots, k_1]$, and $e^T = [e, \dot{e}, \ddot{e}, \cdots, e^{(n-1)}]^T$ are chosen such that the polynomial $e^{(n)} + k_1 e^{(n-1)} + k_2 e^{(n-2)} + \cdots + k_n e$ is Hurwitz. The substitution of the control law of Equation (63) in Equation (61) results into

$$\begin{gathered}
x^{(n)} = f(x,t) + g(x,t)\tfrac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + \tilde{d} \Rightarrow \\
x^{(n)} = f(x,t) + \{\hat{g}(x,t) + [g(x,t) - \hat{g}(x,t)]\}\tfrac{1}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + \tilde{d} \Rightarrow \\
x^{(n)} = f(x,t) + \{\tfrac{\hat{g}(x,t)}{\hat{g}(x,t)}[x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d} \Rightarrow \\
x^{(n)} = f(x,t) + x_m^{(n)} - \hat{f}(x,t) - K^T e + u_c + [g(x,t) - \hat{g}(x,t)]u + u_c + \tilde{d} \Rightarrow \\
x^{(n)} - x_m^{(n)} = -K^T e + [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + u_c + \tilde{d} \Rightarrow \\
x^{(n)} = -K^T e + u_c + [f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d}
\end{gathered} \tag{64}$$

The above relation can be written in a state-equations form. The state vector is taken to be $e^T = [e, \dot{e}, \cdots, e^{(n-1)}]$, which yields

$$\dot{e} = Ae - BK^T e + Bu_c + B\{[f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d}\} \tag{65}$$

or equivalently

$$\begin{aligned} &\dot{e} = (A - BK^T)e + Bu_c + B\{[f(x,t) - \hat{f}(x,t)] + [g(x,t) - \hat{g}(x,t)]u + \tilde{d}\} \\ &e_1 = C^T e \end{aligned} \tag{66}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \\ B^T &= \left(0, 0, \cdots, 0, 1\right), \; C^T = \left(1, 0, \cdots, 0, 0\right) \\ K^T &= \left(k_0, k_1, \cdots, k_{n-2}, k_{n-1}\right) \end{aligned} \tag{67}$$

where e_1 denotes the output error $e_1 = x - x_m$. Equation (66) describes a regulation problem.

4.3. Estimation of the state vector

The control of the Uzawa-Lucas growth model described by Equation (61) becomes more complicated when the state vector x is not directly measurable and has to be reconstructed through a state observer. The following definitions are used

- error of the state vector $e = x - x_m$
- error of the estimated state vector $\hat{e} = \hat{x} - x_m$
- observation error $\tilde{e} = e - \hat{e} = (x - x_m) - (\hat{x} - x_m)$

When an observer is used to reconstruct the state vector, the control law of Equation (63) is written as

$$u = \frac{1}{\hat{g}(\hat{x},t)}[x_m^{(n)} - \hat{f}(\hat{x},t) + K^T e + u_c] \tag{68}$$

Applying Equation (68) to the nonlinear system described by Equation (61), after some operations results into

$$\begin{aligned} x^{(n)} = x_m^{(n)} - K^T \hat{e} + u_c + [f(x,t) - \hat{f}(\hat{x},t)] + \\ [g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d} \end{aligned}$$

It holds $e = x - x_m \Rightarrow x^{(n)} = e^{(n)} + x_m^{(n)}$. Substituting $x^{(n)}$ in the above equation gives

$$\begin{aligned} e^{(n)} + x_m^{(n)} = x_m^{(n)} - K^T \hat{e} + u_c + [f(x,t) - \hat{f}(\hat{x},t)] + \\ + [g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d} \Rightarrow \end{aligned} \tag{69}$$

$$\begin{aligned} \dot{e} = Ae - BK^T \hat{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x},t)] + \\ + [g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d}\} \end{aligned} \tag{70}$$

$$e_1 = C^T e \tag{71}$$

where $e = [e, \dot{e}, \ddot{e}, \cdots, e^{(n-1)}]^T$, and $\hat{e} = [\hat{e}, \dot{\hat{e}}, \ddot{\hat{e}}, \cdots, \hat{e}^{(n-1)}]^T$.

The state observer is designed according to Equations (70) and (71) and is given by [28]:

$$\dot{\hat{e}} = A\hat{e} - BK^T\hat{e} + K_o[e_1 - C^T\hat{e}] \tag{72}$$

$$\hat{e}_1 = C^T\hat{e} \tag{73}$$

The observation gain $K_o = [k_{o_0}, k_{o_1}, \cdots, k_{o_{n-2}}, k_{o_{n-1}}]^T$ is selected so as to assure the convergence of the observer.

4.4. The additional control term u_c

The additional term u_c which appeared in Equation (63) is also introduced in the observer-based control to compensate for:

- The external disturbances $\tilde{d}$
- The state vector estimation error $\tilde{e} = e - \hat{e} = x - \hat{x}$
- The approximation error of the nonlinear functions $f(x,t)$ and $g(x,t)$, denoted as $w = [f(x,t) - \hat{f}(\hat{x},t)] + [g(x,t) - \hat{g}(\hat{x},t)]u$

The control signal u_c consists of 2 terms, namely:

- the H_∞ control term, $u_a = -\frac{1}{r}B^T P\tilde{e}$ for the compensation of d and w
- the control term u_b for the compensation of the observation error $\tilde{e}$

4.5. Dynamics of the observation error

The observation error is defined as $\tilde{e} = e - \hat{e} = x - \hat{x}$. Subtracting Equation (72) from Equation (70) as well as Equation (73) from Equation (71) one gets

$$\begin{aligned}
&\dot{e} - \dot{\hat{e}} = A(e - \hat{e}) + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x},t)] + \\
&+[g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d}\} - K_oC^T(e - \hat{e}) \\
&e_1 - \hat{e}_1 = C^T(e - \hat{e})
\end{aligned}$$

i.e.

$$\begin{gathered}
\dot{\tilde{e}} = A\tilde{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x},t)] + \\
+[g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d}\} - K_oC^T\tilde{e} \\
\tilde{e}_1 = C^T\tilde{e}
\end{gathered}$$

which can be written as

$$\dot{\tilde{e}} = (A - K_oC^T)\tilde{e} + Bu_c + B\{[f(x,t) - \hat{f}(\hat{x},t)] + [g(x,t) - \hat{g}(\hat{x},t)]u + \tilde{d}\} \tag{74}$$

$$\tilde{e}_1 = C\tilde{e} \tag{75}$$

4.6. Approximation of the functions $f(x,t)$ and $g(x,t)$

Neurofuzzy networks can been trained on-line to approximate parts of the dynamic equation of non-linear systems, or to compensate for external disturbances. The approximation of functions $f(x,t)$ and $g(x,t)$

of Equation (61) can be carried out with Takagi-Sugeno neuro-fuzzy networks of zero or first order (Figure 1). These consist of rules of the form:

R^l : IF $\hat{x}$ is A_1^l AND $\dot{\hat{x}}$ is A_2^l AND $\cdots$ AND $\hat{x}^{(n-1)}$ is A_n^l THEN $\bar{y}^l = \sum_{i=1}^n w_i^l \hat{x}_i + b^l$, $l = 1, 2, \cdots, L$

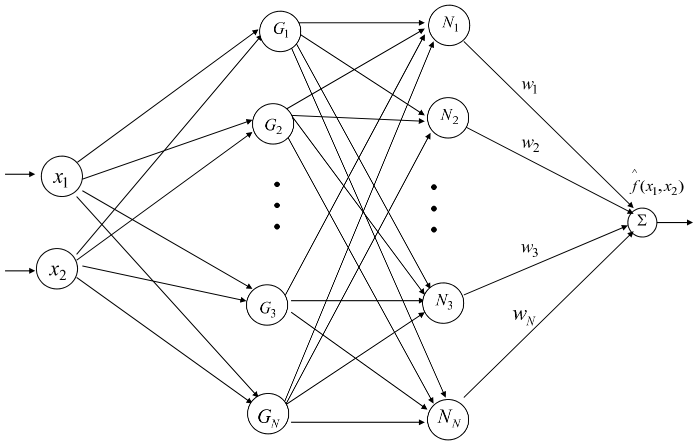


Figure 1. Neuro-fuzzy approximator: G_i Gaussian basis function, N_i: normalization unit.

The output of the neuro-fuzzy model is calculated by taking the average of the consequent part of the rules

$$\hat{y} = \frac{\sum_{l=1}^L \bar{y}^l \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}{\sum_{l=1}^L \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)} \tag{76}$$

where $\mu_{A_i^l}$ is the membership function of x_i in the fuzzy set A_i^l. The training of the neuro-fuzzy networks is carried out with 1^{st} order gradient algorithms, in pattern mode, i.e. by processing only one data pair (x_i, y_i) at every time step i. The estimation of $f(x,t)$ and $g(x,t)$ can be written as

$$\begin{aligned} \hat{f}(\hat{x}|\theta_f) &= \theta_f^T \phi(\hat{x}) \\ \hat{g}(\hat{x}|\theta_g) &= \theta_g^T \phi(\hat{x}) \end{aligned} \tag{77}$$

where $\phi(\hat{x})$ are kernel functions with elements $\phi^l(\hat{x}) = \frac{\prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}{\sum_{l=1}^L \prod_{i=1}^n \mu_{A_i^l}(\hat{x}_i)}$ $l = 1, 2, \cdots, L$. It is assumed that that the weights θ_f and θ_g vary in the bounded areas M_{θ_f} and M_{θ_g} which are defined as

$$\begin{aligned} M_{\theta_f} &= \{\theta_f \in R^h : ||\theta_f|| \leq m_{\theta_f}\} \\ M_{\theta_g} &= \{\theta_g \in R^h : ||\theta_g|| \leq m_{\theta_g}\} \end{aligned} \tag{78}$$

with m_{θ_f} and m_{θ_g} positive constants. The values of θ_f and θ_g for which optimal approximation is succeeded are:

$$\begin{aligned} \theta_f^* &= arg\ min_{\theta_f \in M_{\theta_f}} [sup_{x \in U_x, \hat{x} \in U_{\hat{x}}} |f(x) - \hat{f}(\hat{x}|\theta_f)|] \\ \theta_g^* &= arg\ min_{\theta_g \in M_{\theta_g}} [sup_{x \in U_x, \hat{x} \in U_{\hat{x}}} |g(x) - \hat{g}(\hat{x}|\theta_g)|] \end{aligned}$$

The variation ranges of x and $\hat{x}$ are the compact sets

$$\begin{aligned} U_x &= \{x \in R^n : ||x|| \leq m_x < \infty\}, \\ U_{\hat{x}} &= \{\hat{x} \in R^n : ||\hat{x}|| \leq m_{\hat{x}} < \infty\} \end{aligned} \tag{79}$$

The approximation error of $f(x,t)$ and $g(x,t)$ is given by

$$
\begin{aligned}
&w = [\hat{f}(\hat{x}|\theta_f^*) - f(x,t)] + [\hat{g}(\hat{x}|\theta_g^*) - g(x,t)]u \Rightarrow \\
&w = \{[\hat{f}(\hat{x}|\theta_f^*) - f(x|\theta_f^*)] + [f(x|\theta_f^*) - f(x,t)]\} + \\
&\{[\hat{g}(\hat{x}|\theta_g^*) - g(\hat{x}|\theta_g^*)] + [g(\hat{x}|\theta_g^*)g(x,t)]\}u
\end{aligned}
\tag{80}
$$

where

- $\hat{f}(\hat{x}|\theta_f^*)$ is the approximation of f for the best estimation θ_f^* of the weights' vector θ_f.
- $\hat{g}(\hat{x}|\theta_g^*)$ is the approximation of g for the best estimation θ_g^* of the weights' vector θ_g.

The approximation error w can be decomposed into w_a and w_b, where

$$
\begin{aligned}
w_a &= [\hat{f}(\hat{x}|\theta_f) - \hat{f}(\hat{x}|\theta_f^*)] + [\hat{g}(\hat{x}|\theta_g) - \hat{g}(\hat{x}|\theta_g^*)]u \\
w_b &= [\hat{f}(\hat{x}|\theta_f^*) - f(x,t)] + [\hat{g}(\hat{x}|\theta_g^*) - g(x,t)]u
\end{aligned}
$$

Finally, the following two parameters are defined:

$$
\tilde{\theta}_f = \theta_f - \theta_f^*, \quad \tilde{\theta}_g = \theta_g - \theta_g^* \tag{81}
$$

5. Lyapunov stability analysis

5.1. Design of the Lyapunov function

The adaptation law of the neurofuzzy approximators'weights θ_f and θ_g as well as of the supervisory control term u_c are derived from the requirement for negative definiteness of the Lyapunov function

$$
V = \frac{1}{2}\hat{e}^T P_1 \hat{e} + \frac{1}{2}\tilde{e}^T P_2 \tilde{e} + \frac{1}{2\gamma_1}\tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2\gamma_2}\tilde{\theta}_g^T \tilde{\theta}_g \tag{82}
$$

The selection of the Lyapunov function is based on the following principle of indirect adaptive control $\hat{e}: \lim_{t\to\infty}\hat{x}(t) = x_d(t)$ and $\tilde{e}: \lim_{t\to\infty}\hat{x}(t) = x(t)$. This yields $\lim_{t\to\infty}x(t) = x_d(t)$. Substituting Equations (70), (71) and Equations (74), (75) into Equation (82) and differentiating results into

$$
\dot{V} = \tfrac{1}{2}\dot{\hat{e}}^T P_1 \hat{e} + \tfrac{1}{2}\hat{e}^T P_1 \dot{\hat{e}} + \tfrac{1}{2}\dot{\tilde{e}}^T P_2 \tilde{e} + \tfrac{1}{2}\tilde{e}^T P_2 \dot{\tilde{e}} + \tfrac{1}{\gamma_1}\tilde{\theta}_f^T \dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T \dot{\tilde{\theta}}_g \tag{83}
$$

which in turn gives

$$
\begin{gathered}
\dot{V} = \tfrac{1}{2}\{(A - BK^T)\hat{e} + K_o C^T \tilde{e}\}^T P_1 \hat{e} + \tfrac{1}{2}\hat{e}^T P_1 \{(A - BK^T)\hat{e} + K_o C^T \tilde{e}\} + \\
+\tfrac{1}{2}\{(A - K_o C^T)\tilde{e} + Bu_c + Bd + Bw\}^T P_2 \tilde{e} + \tfrac{1}{2}\tilde{e}^T P_2 \{(A - K_o C^T)\tilde{e} + Bu_c + Bd + Bw\} + \\
+\tfrac{1}{\gamma_1}\tilde{\theta}_f^T \dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T \dot{\tilde{\theta}}_g
\end{gathered}
\tag{84}
$$

or, equivalently

$$
\begin{gathered}
\dot{V} = \tfrac{1}{2}\{\hat{e}^T (A - BK^T)^T + \tilde{e}^T C K_o^T\} P_1 \hat{e} + \tfrac{1}{2}\hat{e}^T P_1 \{(A - BK^T)\hat{e} + K_o C^T \tilde{e}\} + \\
+\tfrac{1}{2}\{\tilde{e}^T (A - K_o C^T)^T + B^T u_c + B^T w + B^T d\} P_2 \tilde{e} + \tfrac{1}{2}\tilde{e}^T P_2 \{(A - K_o C^T)\tilde{e} + Bu_c + Bw + Bd\} + \\
+\tfrac{1}{\gamma_1}\tilde{\theta}_f^T \dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T \dot{\tilde{\theta}}_g
\end{gathered}
\tag{85}
$$

$$
\begin{gathered}
\dot{V} = \tfrac{1}{2}\hat{e}^T (A - BK^T)^T P_1 \hat{e} + \tfrac{1}{2}\tilde{e}^T C K_o^T P_1 \hat{e} + +\tfrac{1}{2}\hat{e}^T P_1 (A - BK^T)\hat{e} + \tfrac{1}{2}\hat{e}^T P_1 K_o C^T \tilde{e} + \\
+\tfrac{1}{2}\tilde{e}^T (A - K_o C^T)^T P_2 \tilde{e} + \tfrac{1}{2}B^T P_2 \tilde{e}(u_c + w + d) + \tfrac{1}{2}\tilde{e}^T P_2 (A - K_o C^T)\tilde{e} + \tfrac{1}{2}\tilde{e}^T P_2 B(u_c + w + d) + \\
+\tfrac{1}{\gamma_1}\tilde{\theta}_f^T \dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T \dot{\tilde{\theta}}_g
\end{gathered}
\tag{86}
$$

Assumption 1: For given positive definite matrices Q_1 and Q_2 there exist positive definite matrices P_1 and P_2, which are the solution of the following Riccati Equation [28]

$$(A-BK^T)^T P_1 + P_1(A-BK^T) + Q_1 = 0 \tag{87}$$

$$\begin{gathered}(A-K_oC^T)^T P_2 + P_2(A-K_oC^T)- \\ -P_2B(\tfrac{2}{r}-\tfrac{1}{\rho^2})B^T P_2 + Q_2 = 0\end{gathered} \tag{88}$$

The conditions given in Equations (87) to (88) are related to the requirement that the systems described by Equations (72), (73) and Equations (74), (75) are strictly positive real. Substituting Equations (87) to (88) into $\dot{V}$ yields

$$\begin{gathered}\dot{V} = \tfrac{1}{2}\hat{e}^T\{(A-BK^T)^T P_1 + P_1(A-BK^T)\}\hat{e} + \tilde{e}^T CK_o^T P_1\hat{e} + \\ +\tfrac{1}{2}\tilde{e}^T\{(A-K_oC^T)^T P_2 + P_2(A-K_oC^T)\}\tilde{e} + B^T P_2\tilde{e}(u_c+w+d) + \\ +\tfrac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{gathered} \tag{89}$$

which is also written as

$$\begin{gathered}\dot{V} = -\tfrac{1}{2}\hat{e}^T Q_1\hat{e} + \tilde{e}^T CK_o^T P_1\hat{e} - \tfrac{1}{2}\tilde{e}^T\{Q_2 - P_2B(\tfrac{2}{r}-\tfrac{1}{\rho^2})B^T P_2\}\tilde{e} + B^T P_2\tilde{e}(u_c+w+d) + \\ +\tfrac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{gathered} \tag{90}$$

The supervisory control u_c is decomposed in two terms, u_a and u_b

$$u_a = -\frac{1}{r}p_{1n}\tilde{e}_1 = -\frac{1}{r}\tilde{e}^T P_2B + \frac{1}{r}(p_{2n}\tilde{e}_2 + \cdots + p_{nn}\tilde{e}_n) = -\frac{1}{r}\tilde{e}^T P_2B + \Delta u_a \tag{91}$$

where p_{1n} stands for the last (n-th) element of the first row of matrix P_2, and

$$u_b = -[(P_2B)^T(P_2B)]^{-1}(P_2B)^T CK_o^T P_1\hat{e} \tag{92}$$

- u_a is an H_∞ control used for the compensation of the approximation error w and the additive disturbance $\tilde{d}$. Its first component $-\frac{1}{r}\tilde{e}^T P_2B$ has been chosen so as to compensate for the term $\frac{1}{r}\tilde{e}^T P_2BB^T P_2\tilde{e}$, which appears in Equation (90). By subtracting the second component $-\frac{1}{r}(p_{2n}\tilde{e}_2 + \cdots + p_{nn}\tilde{e}_n)$ one has that $u_a = -\frac{1}{r}p_{1n}\tilde{e}_1$, which means that u_a is computed based on the feedback the measurable variable $\tilde{e}_1$. Equation (91) is finally rewritten as $u_a = -\frac{1}{r}\tilde{e}^T P_2B + \Delta u_a$.
- u_b is a control used for the compensation of the observation error (the control term u_b has been chosen so as to satisfy the condition $\tilde{e}^T P_2Bu_b = -\tilde{e}^T CK_o^T P_1\hat{e}$.

The control scheme is depicted in Figure 2. Substituting Equations (91) and (92) in $\dot{V}$, one gets

$$\begin{gathered}\dot{V} = -\tfrac{1}{2}\hat{e}^T Q_1\hat{e} + \tilde{e}^T CK_o^T P_1\hat{e} - \tfrac{1}{2}\tilde{e}^T Q_2\tilde{e} + \tfrac{1}{r}\tilde{e}^T P_2BB^T P_2\tilde{e} - \\ -\tfrac{1}{2\rho^2}\tilde{e}^T P_2BB^T P_2\tilde{e} + \tilde{e}^T P_2Bu_b - \tfrac{1}{r}\tilde{e}^T P_2BB^T P_2\tilde{e} + B^T P_2\tilde{e}(w+d+\Delta u_a) + \\ +\tfrac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{gathered} \tag{93}$$

or equivalently,

$$\begin{gathered}\dot{V} = -\tfrac{1}{2}\hat{e}^T Q_1\hat{e} - \tfrac{1}{2}\tilde{e}^T Q_2\tilde{e} - \tfrac{1}{2\rho^2}\tilde{e}^T P_2BB^T P_2\tilde{e} + B^T P_2\tilde{e}(w+d+\Delta u_a) + \\ +\tfrac{1}{\gamma_1}\tilde{\theta}_f^T\dot{\tilde{\theta}}_f + \tfrac{1}{\gamma_2}\tilde{\theta}_g^T\dot{\tilde{\theta}}_g\end{gathered} \tag{94}$$

It holds that $\dot{\tilde{\theta}}_f = \dot{\theta}_f - \dot{\theta}_f^* = \dot{\theta}_f$ and $\dot{\tilde{\theta}}_g = \dot{\theta}_g - \dot{\theta}_g^* = \dot{\theta}_g$. The following weight adaptation laws are considered:

$$\dot{\theta}_f = \{ \begin{matrix} -\gamma_1\tilde{e}^T P_2B\phi(\hat{x}) \ \ if \ ||\theta_f|| < m_{\theta_f} \\ 0 \ \ ||\theta_f|| \geq m_{\theta_f} \end{matrix} \tag{95}$$

$$\dot{\theta}_g = \{ \begin{array}{c} -\gamma_2 \tilde{e}^T P_2 B\phi(\hat{x})u_c \;\; if \; ||\theta_g|| < m_{\theta_g} \\ 0 \;\; ||\theta_g|| \geq m_{\theta_g} \end{array} \tag{96}$$

To set $\dot{\theta}_f$ and $\dot{\theta}_g$ equal to 0, when $||\theta_f \geq m_{\theta_f}||$, and $||\theta_g \geq m_{\theta_g}||$ the projection operator is employed [28]:

$$\begin{array}{c} P\{\gamma_1 \tilde{e}^T P_2 B\phi(\hat{x})\} = -\gamma_1 \tilde{e}^T P_2 B\phi(\hat{x}) + \\ +\gamma_1 \tilde{e}^T P_2 B \frac{\theta_f \theta_f^T}{||\theta_f||^2}\phi(\hat{x}) \\ P\{\gamma_1 \tilde{e}^T P_2 B\phi(\hat{x})u_c\} = -\gamma_1 \tilde{e}^T P_2 B\phi(\hat{x})u_c + \\ +\gamma_1 \tilde{e}^T P_2 B \frac{\theta_f \theta_f^T}{||\theta_f||^2}\phi(\hat{x})u_c \end{array}$$

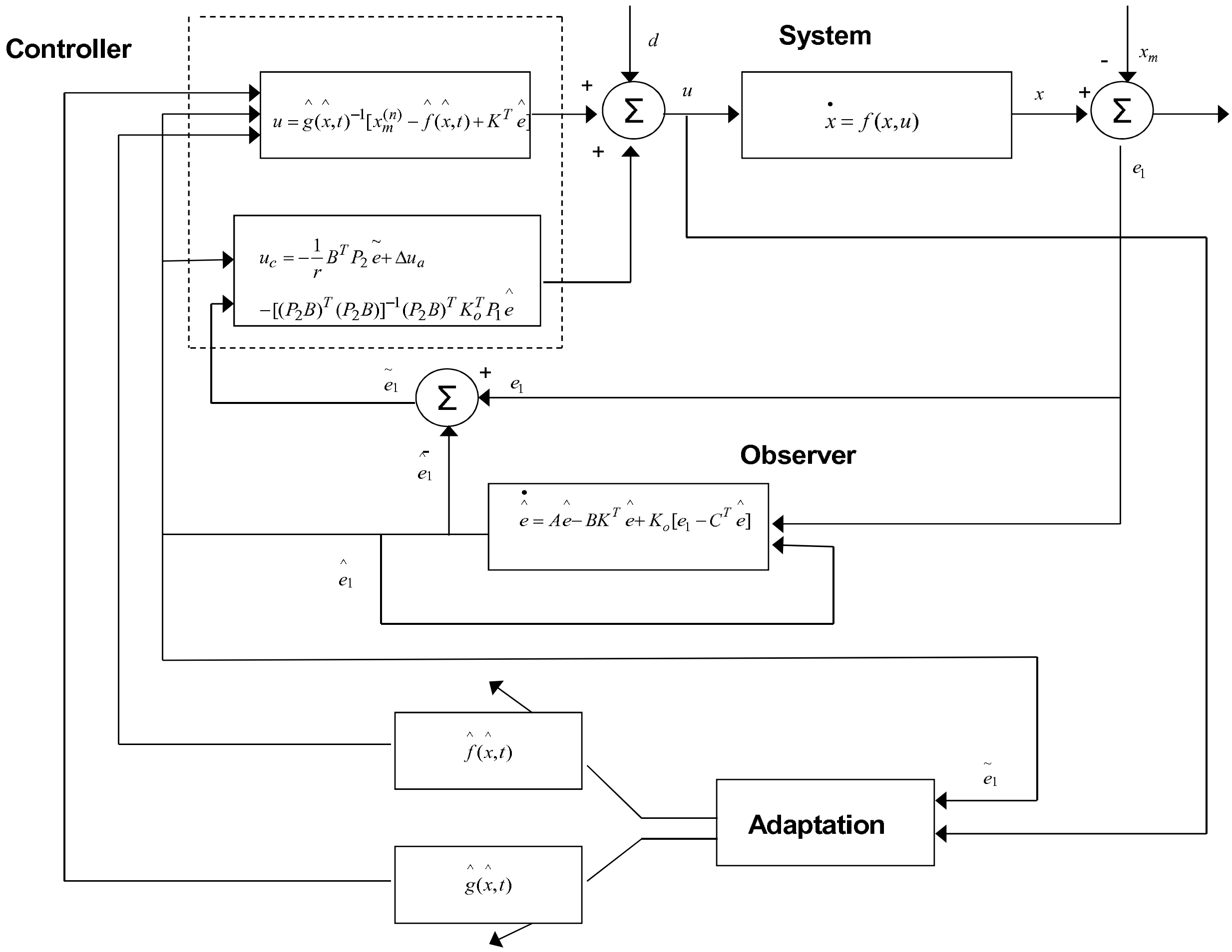


Figure 2. The proposed adaptive neurofuzzy control scheme.

The update of θ_f stems from a gradient algorithm on the cost function $\frac{1}{2}(f - \hat{f})^2$ [32–34]. The update of θ_g is also of the gradient type, while u_c implicitly tunes the adaptation gain γ_2. Substituting Equations (95) and (96) in $\dot{V}$ gives

$$\begin{array}{c} \dot{V} = -\frac{1}{2}\hat{e}^T Q_1 \hat{e} - \frac{1}{2}\tilde{e}^T Q_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} + B^T P_2 \tilde{e}(w + d + \Delta u_a) + \\ +\frac{1}{\gamma_1}\tilde{\theta}_f^T(-\gamma_1 \tilde{e}^T P_2 B\phi(\hat{x})) + \frac{1}{\gamma_2}\tilde{\theta}_g^T(-\gamma_2 \tilde{e}^T P_2 B\phi(\hat{x})u) \end{array} \tag{97}$$

which is also written as

$$\dot{V} = -\frac{1}{2}\hat{e}^T Q_1 \hat{e} - \frac{1}{2}\tilde{e}^T Q_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} + \tilde{e}^T P_2 B(w + d + \Delta u_a) - \\ -\tilde{e}^T P_2 B \tilde{\theta}_f^T \phi(\hat{x}) - \tilde{e}^T P_2 B \tilde{\theta}_g^T \phi(\hat{x}) u \tag{98}$$

and using Equations (77) and (81) results into

$$\dot{V} = -\frac{1}{2}\hat{e}^T Q_1 \hat{e} - \frac{1}{2}\tilde{e}^T Q_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} + \tilde{e}^T P_2 B(w + d + \Delta u_a) - \\ -\tilde{e}^T P_2 B\{[\hat{f}(\hat{x}|\theta_f) + \hat{g}(\hat{x}|\theta_f)u] - [\hat{f}(\hat{x}|\theta_f^*) + \hat{g}(\hat{x}|\theta_g^*)u]\} \tag{99}$$

where $[\hat{f}(\hat{x}|\theta_f) + \hat{g}(\hat{x}|\theta_f)u] - [\hat{f}(\hat{x}|\theta_f^*) + \hat{g}(\hat{x}|\theta_g^*)u] = w_a$. Thus setting $w_1 = w + w_a + d + \Delta u_a$ one gets

$$\dot{V} = -\frac{1}{2}\hat{e}^T Q_1 \hat{e}\frac{1}{2}\tilde{e}^T Q_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} + B^T P_2 \tilde{e} w_1 \Rightarrow \\ \dot{V} = -\frac{1}{2}\hat{e}^T Q_1 \hat{e}\frac{1}{2}\tilde{e}^T Q_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} + \frac{1}{2}w_1^T B^T P_2 \tilde{e} + \frac{1}{2}\tilde{e}^T P_2 B w_1 \tag{100}$$

Lemma: The following inequality holds

$$\frac{1}{2}\tilde{e}^T P_2 B w_1 + \frac{1}{2}w_1^T B^T P_2 \tilde{e} - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} \le \frac{1}{2}\rho^2 w_1^T w_1 \tag{101}$$

Proof: The binomial $(\rho a - \frac{1}{\rho}b)^2 \ge 0$ is considered. Expanding the left part of the above inequality one gets

$$\rho^2 a^2 + \frac{1}{\rho^2}b^2 - 2ab \ge 0 \Rightarrow \frac{1}{2}\rho^2 a^2 + \frac{1}{2\rho^2}b^2 - ab \ge 0 \\ \Rightarrow ab - \frac{1}{2\rho^2}b^2 \le \frac{1}{2}\rho^2 a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \le \frac{1}{2}\rho^2 a^2 \tag{102}$$

The following substitutions are carried out: $a = w_1$ and $b = \tilde{e}^T P_2 B$ and the previous relation becomes

$$\frac{1}{2}w_1^T B^T P_2 \tilde{e} + \frac{1}{2}\tilde{e}^T P_2 B w_1 - \frac{1}{2\rho^2}\tilde{e}^T P_2 B B^T P_2 \tilde{e} \\ \le \frac{1}{2}\rho^2 w_1^T w_1 \tag{103}$$

The above inequality is used in $\dot{V}$, and the right part of the associated inequality is enforced

$$\dot{V} \le -\frac{1}{2}\hat{e}^T Q_1 \hat{e} - \frac{1}{2}\tilde{e}^T Q_2 \tilde{e} + \frac{1}{2}\rho^2 w_1^T w_1 \tag{104}$$

Thus, Equation (104) can be written as

$$\dot{V} \le -\frac{1}{2}E^T Q E + \frac{1}{2}\rho^2 w_1^T w_1 \tag{105}$$

where

$$E = \begin{pmatrix} \hat{e} \\ \tilde{e} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} = diag[Q_1, Q_2] \tag{106}$$

Hence, the H_∞ performance criterion is derived. For ρ sufficiently small Equation (104) will be true and the H_∞ tracking criterion will be satisfied. In that case, the integration of $\dot{V}$ from 0 to T gives

$$\int_0^T \dot{V}(t)dt \le -\frac{1}{2}\int_0^T ||E||^2 dt + \frac{1}{2}\rho^2 \int_0^T ||w_1||^2 dt \Rightarrow \\ 2V(T) - 2V(0) \le -\int_0^T ||E||_Q^2 dt + \rho^2 \int_0^T ||w_1||^2 dt \Rightarrow \\ 2V(T) + \int_0^T ||E||_Q^2 dt \le 2V(0) + \rho^2 \int_0^T ||w_1||^2 dt$$

It is assumed that there exists a positive constant $M_w > 0$ such that $\int_0^\infty ||w_1||^2 dt \le M_w$. Therefore for the integral $\int_0^T ||E||_Q^2 dt$ one gets

$$\int_0^\infty ||E||_Q^2 dt \le 2V(0) + \rho^2 M_w \tag{107}$$

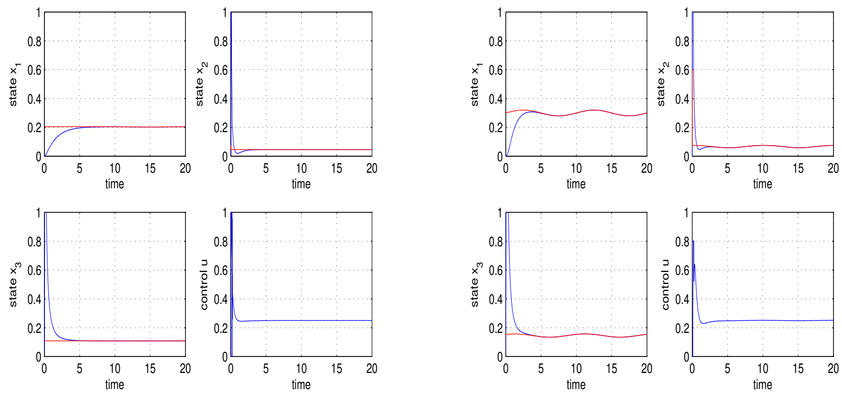
Thus, the integral $\int_0^\infty ||E||_Q^2 dt$ is bounded and according to Barbalat's Lemma

$$\lim_{t\to\infty} E(t) = 0 \Rightarrow \begin{array}{l} \lim_{t\to\infty} \hat{e}(t) = 0 \\ \lim_{t\to\infty} \tilde{e}(t) = 0 \end{array}$$

Therefore $\lim_{t\to\infty} e(t) = 0$.

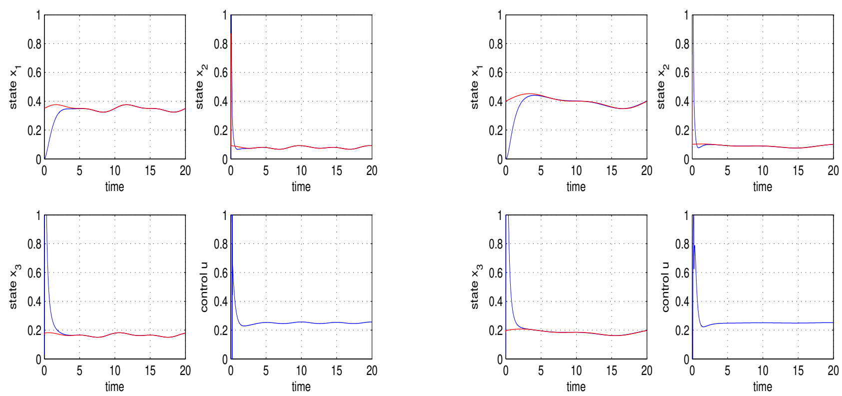
6. Simulation tests

The proposed adaptive fuzzy control method has been applied to the problem of stabilization of the dynamics of the Uzawa-Lucas endogenous growth model and its performance has been checked through simulation experiments in the case of tracking of several reference trajectories. The presented results are depicted in Figure 3 to Figure 5. It has been confirmed that all state variables converged fast to the reference trajectories and that the tracking error was minimized. Moreover,the control inputs computed by the adaptive neurofuzzy controller varied smoothly.



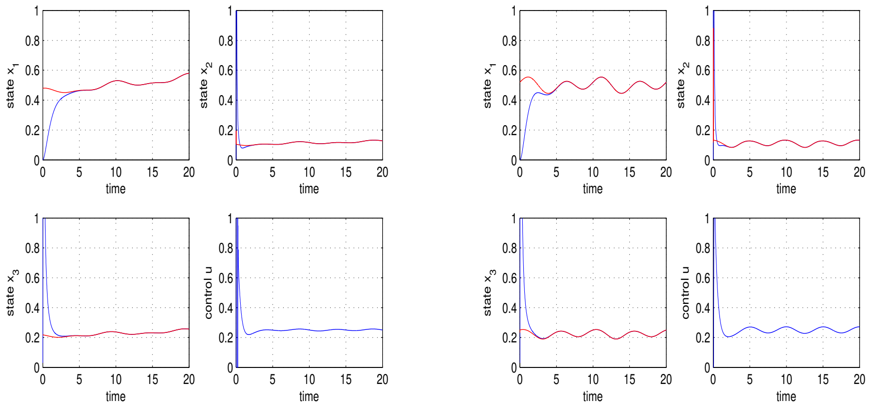
(a) **(b)**

Figure 3. Tracking of setpoints (red lines) by the state variables x_1, x_2, x_3 (blue lines) of the Uzawa-Lucas endogenous growth model and variations of the control input u under flateess-based adaptive fuzzy control: **(a)** Test case 1; **(b)** Test case 2.



(a) **(b)**

Figure 4. Tracking of setpoints (red lines) by the state variables x_1, x_2, x_3 (blue lines) of the Uzawa-Lucas endogenous growth model and variations of the control input u under flateess-based adaptive fuzzy control: **(a)** Test case 3; **(b)** Test case 4.



(a) **(b)**

Figure 5. Tracking of setpoints (red lines) by the state variables x_1,x_2,x_3 (blue lines) of the Uzawa-Lucas endogenous growth model and variations of the control input u under flateess-based adaptive fuzzy control: **(a)** Test case 5; **(b)** Test case 6.

The approximators' inputs were the system's state variables $x_1 = m$, $x_2 = g$ and $x_3 = \varepsilon$. Knowing that there are $i = 3$ state variables for the chaotic finance system and that each such variable comprises $n = 3$ fuzzy sets, the total number of rules in the fuzzy rule base should be $n^m = 3^3 = 27$. The aggregate output of the neuro-fuzzy approximator (rule-base) for function $f(x)$ is given by Equation (76). The membership functions in the fuzzy rules have been Gaussians functions. The centers $c_i^{(l)}$, $i = 1,\cdots,3$ and the variances $v^{(l)}$ of each rule are summarized in Table 1. Similar is the structure of the neuro-fuzzy approximator for function $g(x)$.

Table 1. Parameters of the fuzzy rule base.

Rule	$c_1^{(l)}$	$c_2^{(l)}$	$c_3^{(l)}$	$v^{(l)}$
$R^{(1)}$	-1.0	-1.0	-1.0	3
$R^{(2)}$	-1.0	-1.0	0.0	3
$R^{(3)}$	-1.0	-1.0	1.0	3
$R^{(4)}$	-1.0	0.0	-1.0	3
$R^{(5)}$	-1.0	0.0	0.0	3
$R^{(6)}$	-1.0	0.0	1.0	3
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$R^{(27)}$	1.0	1.0	1.0	3

The control loop was based on simultaneous estimation of the unknown endogenous growth model (this was performed with the use of neuro-fuzzy approximators) and of the nonmeasurable elements of the growth model's state vector, that is of state variable x_2 and of state variable x_3 (this was performed with the use of the state observer). The design details of the neurofuzzy approximators and particularly the weights' update rates that ensure convergence of the learning process have been defined in Equation (95) and Equation (96). The obtained results are depicted in Figure 3 to Figure 5. The diagrams present rated (normalized) values for both the state variables and the control inputs, in the interval $[0, 1]$. The state variables x_1, x_2 and x_3, as well as the control input u are printed in blue colour, while the related setpoints are printed in red colour.

The proposed adaptive fuzzy control method is a model-free one. This means that it needs no prior knowledge about the Uzawa-Lucas growth model. Moreover it is implemented only with the use of output feedback. The control scheme provides simultaneously solution to three optimization problems: (i) minimization of the state vector's tracking error, which is the distance of the state vector from the reference trajectories, (ii) minimization of the state-vector's estimation error, that is the state vector computed by the state observer finally converges to the real value of the system's state vector, (iii) minimization of the estimation error for the unknown dynamics of the financial system being identified with the use of neuro-fuzzy approximators. The control method is applicable after finding a solution for the two Riccati equations given in Equation (87) and Equation (88).

7. Conclusions

The article has proposed a novel flatness-based adaptive fuzzy control method, that relies exclusively on output feedback, for the Uzawa-Lucas endogenous growth model. The necessity for applying adaptive control is due to the imprecision of this state-space model and due to the unknown or changing values of the model's parameters. It has been proven that the model is differentially flat. This signifies that through the application of a change of state variables one can transform the initial nonlinear state-space model into an equivalent input-output linearized form, or into the Brunovsky canonical form. Considering the latter description of the system an adaptive fuzzy controller has been developed.

The proposed adaptive fuzzy controller achieves simultaneous estimation of the unknown dynamics of the Uzawa-Lucas model and stabilization of the system, that is convergence of its state variables to the related reference setpoints. The learning of the unknown dynamics was performed through neurofuzzy approximators which were iteratively updated with the use of gradient algorithms. Since only feedback of the system's output was used, state vector estimation was performed with the use of a convergent state observer. The computation of the gains of the adaptive fuzzy controller required the solution of two algebraic Riccati equations. The global stability properties of the control scheme were proven through Lyapunov analysis.

Acknowledgments

Funding support: Ref. No. 5610 State-space approaches for modelling and control in financial engineering of the Unit of Industrial Automation of the Industrial Systems Institute. The authors of the article want to thank other researchers in the subject area for their contribution to the formulation and analysis of endogenous growth models.

Authors' contribution

Conceptualization: G.R., P.S., and F.Z.; methodology: G.R.; software: G.R.; validation: G.R., P.S., and F.Z.; formal analysis: G.R., P.S., and F.Z.; investigation: G.R.; resources: G.R., P.S. and F.Z.; data curation: N/A; writing—original draft preparation: G.R.; writing—review and editing: G.R.; visualization:

G.R.; supervision: G.R.; project administration: G.R.; funding acquisition: G.R. All authors have read and agreed to the published version of the manuscript.

Conflicts of interests

The authors declare no conflict of interest.

References

[1] Barnett WA, Ghosh T. Stability analysis of Uzawa-Lucas endogenous growth model. *Econ. Theory Bull.* 2014, 2(1):33–44.

[2] Barnett WA, Chen G. Bifurcation of macroeconometric models with robustness of dynamical inferences. *Found. Trends Econom.* 2015, 8(1–2):1–144.

[3] Rigatos G, Zouari F, Cuccurullo G, Siano P, Ghosh T. Flatness-based adaptive fuzzy control for the Uzawa-Lucas endogenous growth model. In *International Conference Of Numerical Analysis And Applied Mathematics ICNAAM 2019*, Rhodes, Greece, September 23–28, 2019.

[4] Lucas Jr RE. On the mechanics of economic development. *J. Monet. Econ.* 1988, 22(1):3–42.

[5] LaTorre D, Marsiglia S. Endogenous technological progress in a multi-sector growth model. *Econ. Modell.* 2010, 27(5):1017–1028.

[6] Buccekine R, Ruiz-Tamari JR. Special functions for the study of economic dynamics: the case of the Lucas-Uzawa model. *J. Math. Econ.* 2008, 44(1):33–54.

[7] Ben-Gad M. The two-sector endogenous growth model: an atlas. *J. Macroecon.* 2012, 34(3):706–722.

[8] Hiraguchi R. A note on the closed-form solution of the Lucas-Uzawa model externality. *J. Econ. Dyn. Control* 2009, 33(10):1757–1760.

[9] Chilarescu C. On the existence and uniqueness of solution to the Lucas-Uzawa model. *Econ. Model.* 2011, 28(1–2):109–117.

[10] Gong G, Greiner A, Semmler W. The Uzawa-Lucas model without scale effects: theory and empirical evidence. *Struct. Change Econ. Dyn.* 2004, 15(4):401–420.

[11] Hiraguchi R. A solution to the Lucas-Uzawa model with increasing returns to scale: note. *Econ. Model.* 2009, 26(5):831–834.

[12] Bethmann D. The open-loop solution of the Uzawa-Lucas model of endogenous growth with N agents. *J. Macroecon.* 2008, 30(1):396–414.

[13] Naz R, Chaudhury A, Mohamad EM. Closed-form solution for the Lucas-Uzawa model of economic growth via the partial Hamiltonian approach. *Commun. Nonlinear Sci. Numer. Simul.* 2016, 30(1–3):299–306.

[14] Barnett WA, Ghosh T. Bifurcation analysis of an endogenous growth model. *J. Econ. Asymmetries* 2013, 10(1):53–64.

[15] Arnold LG. A generalized multi-country endogenous growth model. *J. Int. Econ. Econ. Policy* 2007, 4(1):61–100.

[16] Nakano M, Takahashi A, Takahashi S. State-space approach to adaptive fuzzy modelling for financial investment. *Appl. Soft Comput.* 2019, 82:105590.

[17] Gong X, Liu X, Xiong X. Chaotic analysis and adaptive synchronization for a class of fractional-order financial systems. *Physica A* 2019, 522:33–42.

[18] Jajarmi A, Hajipour H, Baleanou B. New aspects of the adaptive synchronization and hyperchaos suppresion of a financial model. *Chaos Solitons Fractals* 2017, 99:285–296.

[19] Wang Z, Huang X, Shi G. Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay. *Comput. Math. Appl.* 2011, 62(3):1531–1539.

[20] Hajipour A, Hajipour M, Baleanu D. On the adaptive sliding-mode controller for a hyperchaotic fractional-order financial system. *Physica A* 2018, 497:139–153.

[21] Gong X, Liu X, Xiong X. Chaotic analysis and adaptive synchronization for a class of fractional-order financial systems *Physica A* 2019, 522:33–42.

[22] Huang C, Cao J. Active control strategy for synchronization and anti-synchronization of a fractional chaotic financial system. *Physica A* 2017, 473:262–275.

[23] Berardi M, Galimberti JK. On the initialization of adaptive learning in macroeconomic models. *J. Econ. Dyn. Control* 2017, 78:26–53.

[24] Cai G, Li Y, Shi W, Zhang Z, Feng G. Synchronization of the delay financial hyperchaotic system via a single adaptive feedback controller. In *2017 13th International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery (ICNC-FSKD)*, Guilin, China, July 29–31, 2017, pp. 388–393.

[25] Hu C, Yu J. Generalized intermittent control and its adaptive strategy on stabilization and synchronization of chaotic systens. *Chaos, Solitons Fractals* 2016, 91:262–269.

[26] Wang Z, Huang X, Shen H. Control of uncertain fractional-order economic system via adaptive sliding-mode. *Neurocomputing* 2012, 83:83–88.

[27] Rigatos G. *State-space approaches for modelling and control in financial engineering: Systems theory and machine learning methods*, 1st ed. Cham: Springer International Publishing, 2017.

[28] Rigatos G. *Nonlinear control and filtering using differential flatness approaches: applications to electromechanicsl systems*, 1st ed. Cham: Springer International Publishing, 2015.

[29] Rudolph J. *Flatness Based Control of Distributed Parameter Systems, Examples and Computer Exercises from Various Technological Domains*, 1st ed. Aachen: Shaker Verlag GmbH, 2003.

[30] Fliess M, Mounier H. Tracking control and π-freeness of infinite dimensional linear systems. In *Progress in Systems and Control Theory*, 1st ed. Basel: Birkhäuser Verlag, 1999. pp. 45–68.

[31] Sira-Ramirez H, Agrawal S. *Differentially Flat Systems*, 1st ed. New York: Marcel Dekker, 2004.

[32] Rigatos GG, Tzafestas SG. Extended Kalman filtering for fuzzy modelling and multi-sensor fusion. *Math. Comput. Modell. Dyn. Syst.* 2007, 13(3):251–266.

[33] Basseville M, Nikiforov I. *Detection of abrupt changes: Theory and Applications*, 1st ed. New Jersey: Prentice-Hall, 1993.

[34] Rigatos G, Zhang Q. Fuzzy model validation using the local statistical approach. *Fuzzy Sets Syst.* 2009, 160(7):882–904.