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A new mathematical strategy for creating asymmetric continuous distributions

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Abstract: In this article, we present a new mathematical strategy for creating flexible asymmetric continuous distributions. It is designed to introduce asymmetry into any distribution with the entire real line as support, thanks to the tuning of two parameters and an intermediate function. A wide range of intermediate functions of different types can be chosen, including a high degree of adaptability. To illustrate this strategy, we present four types of asymmetric normal distributions and four types of asymmetric Cauchy distributions. Some of them have rare properties, such as multimodality (bimodality, trimodality, and more) and abrupt angular shape for the corresponding probability density functions. These features are supported by an extensive graphical analysis. Finally, we discuss the adaptation of the strategy for distributions with different support.

Keywords: continuous distributions; asymmetry; quantile function; normal distribution; Cauchy distribution

1. Introduction

In probability and statistics, developing adaptable models to analyze various types of data is essential. Traditional symmetric distributions, such as the normal and Cauchy distributions, often fail to capture the inherent asymmetries present in real-world data. For example, in finance, asset returns often exhibit asymmetry that symmetric distributions cannot account for. Similarly, in environmental studies, some measurements may be naturally skewed due to physical constraints or underlying processes.

Several mathematical strategies have been proposed in the literature to address this limitation. The two most well-known are those of [1, 2]. The strategy proposed in [2], in particular, has been the focus of extensive research. Its mathematical basis can be described as follows: Let us consider a continuous baseline distribution with support on the entire real line, i.e., $\mathbb{R}$, cumulative distribution function (cdf) denoted by F, and probability density function (pdf) denoted by f. We suppose that this distribution is symmetric around 0, meaning that $f(-x) = f(x)$ for any $x \in \mathbb{R}$. Then the strategy by Azzalini in [2] suggests the asymmetric distribution defined by the following pdf:

$$g(x) = 2f(x)F(ax), \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$. In particular, by choosing the normal distribution for the baseline distribution, g defines the skewed normal distribution, which has received much attention and various extensions. For related work, see [3–11]. Other asymmetric normal distributions resulting from different strategies can be found in [12–17]. See also the book [18] and its references. Besides the asymmetric normal distribution, notable cases include the skewed Cauchy distribution developed in [19], the skewed Student distribution introduced in [20], the skewed logistic

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distribution proposed in [21], the skewed Laplace distribution examined in [22], and the skewed hyperbolic secant distribution elaborated in [23], among others.

In this article, we make a theoretical contribution to this topic. Specifically, we propose a new mathematical strategy that introduces asymmetry into any continuous distribution with support $\mathbb{R}$. This strategy differs from others by exploiting the flexibility of two tunable parameters (with almost no constraints on them) and an intermediate function. This intermediate function can be of various types, such as an exponential function, a power function, a trigonometric function or a logarithmic function. These adjustable settings allow many levels of asymmetry to be achieved. In particular, extremely asymmetric angular or abrupt cases and original multimodality of the oscillatory type are observed in the functions involved, beyond the possibilities of those of the standard asymmetric distributions. We thus respond to the need for more adaptive models that can better fit data with skewed characteristics. The first part presents the main theory, focusing on the quantile properties. The second part demonstrates the application of our strategy by constructing four types of asymmetric normal distributions and four types of asymmetric Cauchy distributions. These illustrate the versatility and practical utility of our approach. We provide a comprehensive graphical analysis to visualize the impact of the introduced asymmetry. Some possible extensions to distributions with different support are also discussed.

The article is structured as follows: Section 2 describes our strategy. Section 3 presents the new asymmetric distributions derived from the normal and Cauchy distributions. Some complementary results are given in Section 4. Section 5 concludes with a discussion of the implications and potential applications of our findings. An appendix containing the proofs of the main results can be found in Appendix.

2. Strategy to construct asymmetric continuous distributions

This section presents our strategy for constructing asymmetric continuous distributions, gives some examples, and develops the main related theory, including quantile and moment analysis.

2.1. *Description*

Our main strategy is described in the proposition below.

Proposition 2.1 *Let us consider a continuous distribution with support $\mathbb{R}$, cdf denoted by F and pdf denoted by f. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a \neq b$. Let ℓ be a differentiable increasing function on $\mathbb{R}$ such that $\lim_{x\to-\infty} \ell(x) = 0$ and $\lim_{x\to+\infty} \ell(x) = +\infty$. Let us now set*

$$G(x) = \frac{1}{F(b) - F(a)} \left\{ F\left[\frac{a + b\ell(x)}{1 + \ell(x)}\right] - F(a) \right\}, \quad x \in \mathbb{R}. \tag{1}$$

Then G is a valid cdf that defines a distribution with support on $\mathbb{R}$.

As mentioned above, the proof of this and all future results is deferred to the Appendix.

The cdf in Equation (1) can also be written as the following comprehensible form:

$$G(x) = \frac{1}{F(b) - F(a)} \{F[b + m(x)] - F(a)\}, \quad x \in \mathbb{R},$$

where

$$m(x) = \frac{a - b}{1 + \ell(x)}.$$

Looking at this formula, we notice some similarity to the general expression for the cdf of the truncated distributions, except that G is for a distribution with support $\mathbb{R}$. See [24]. Here, a and b are not related to the support; they define tuning parameters that only affect the main functions.

Let us now discuss the excluded critical case $a = b$. Setting $b = a + \varepsilon$, we can write

$$G(x) = \frac{\varepsilon}{F(a+\varepsilon) - F(a)} \frac{F\{a + \varepsilon\ell(x)/[1+\ell(x)]\} - F(a)}{\varepsilon\ell(x)/[1+\ell(x)]} \frac{\ell(x)}{1+\ell(x)}.$$

Using standard limit results with $F' = f$, we get

$$\lim_{\varepsilon\to 0} G(x) = \frac{1}{f(a)} f(a) \frac{\ell(x)}{1+\ell(x)} = \frac{\ell(x)}{1+\ell(x)}.$$

We can prove that this function is a valid cdf. Thus, with this development, for the special case $a = b$, the resulting cdf does not depend on the baseline distribution and is equal to $\ell(x)/[1+\ell(x)]$.

A simple technique to avoid the case $a = b$ is to apply a re-parameterisation: we can replace $b = a + \delta$ with $\delta > 0$ or $\delta < 0$, depending on the context.

It follows from the proof of Proposition 2.1 that the pdf associated with G is

$$g(x) = \frac{b-a}{F(b) - F(a)} \frac{\ell'(x)}{[1+\ell(x)]^2} f\left[\frac{a + b\ell(x)}{1+\ell(x)}\right], \quad x \in \mathbb{R}.$$

Of course, if the baseline distribution is symmetric, the distribution defined by G or g is designed to break that symmetry. The asymmetry created can be of different degrees, depending on a, b and the definition of ℓ.

The lemma below describes simple examples of functions ℓ that satisfy the assumptions of Proposition 2.1.

Lemma 2.2 *The following functions ℓ satisfy:* $\lim_{x\to-\infty} \ell(x) = 0$, $\lim_{x\to+\infty} \ell(x) = +\infty$, *$\ell$ is differentiable on $\mathbb{R}$ and ℓ is increasing:*

1. $\ell(x) = \exp(x)$,
2. $\ell(x) = x + \sqrt{x^2+1}$,
3. $\ell(x) = \exp[x + \sin(x)]$,
4. $\ell(x) = \ln[\exp(x) + 1]$.

Of course, there are many more functions possible on a similar basis than those in this lemma. We can mention $\ell(x) = \exp(x)/[1+\exp(-x)]$, $\ell(x) = \exp(x^3)$, $\ell(x) = \exp[x + \cos(x)]$, $\ell(x) = \ln\left[1 + x + \sqrt{x^2+1}\right]$, and all the linear combinations between these functions. For the purposes of this article, however, we will focus on those in the lemma because of their simplicity and diversity in nature. Indeed, they are exponential, power-polynomial, exponential-trigonometric, and logarithmic-exponential, respectively.

2.2. *Theoretical framework*

To complement the descriptive part above, we now present some theory related to the general asymmetric distribution defined by G given by Equation (1), starting with some key functions. The corresponding reliability function is given as

$$\begin{aligned} S(x) &= 1 - G(x) \\ &= 1 - \frac{1}{F(b) - F(a)} \left\{ F\left[\frac{a + b\ell(x)}{1+\ell(x)}\right] - F(a) \right\} \\ &= \frac{1}{F(b) - F(a)} \left\{ F(b) - F\left[\frac{a + b\ell(x)}{1+\ell(x)}\right] \right\}, \quad x \in \mathbb{R}. \end{aligned}$$

From this, we derive the hazard rate function (hrf) as

$$\begin{aligned} h(x) &= \frac{g(x)}{S(x)} \\ &= (b-a)\frac{\ell'(x)}{[1+\ell(x)]^2} f\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]\left\{F(b)-F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]\right\}^{-1}, \quad x \in \mathbb{R}. \end{aligned}$$

We can also present the corresponding cumulative hrf as $H(x) = -\ln[S(x)]$ with $x \in \mathbb{R}$. Let us distinguish $b > a$ and $a > b$. For any $b > a$, we have $F(b) - F(a) > 0$ and

$$H(x) = \ln[F(b)-F(a)] - \ln\left\{F(b)-F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]\right\}, \quad x \in \mathbb{R}.$$

For any $a > b$, we have $F(a) - F(b) > 0$ and

$$H(x) = \ln[F(a)-F(b)] - \ln\left\{F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]-F(b)\right\}, \quad x \in \mathbb{R}.$$

The result below determines the corresponding quantile function (qf), which requires further development.

Proposition 2.3 *Let us consider the context of Proposition 2.1. Then the qf associated with G is given by*

$$Q(y) = \ell^{-1}\left[\frac{a-F^{-1}\{y[F(b)-F(a)]+F(a)\}}{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}\right], \quad y \in (0,1),$$

where F^{-1} is the qf associated with F and ℓ^{-1} is the inverse function of ℓ.

This proposition is of interest because an explicit expression for the qf allows direct calculation of critical values and quartiles: the first quartile is obtained by taking $y = 1/4$, the second quartile (median) is obtained by taking $y = 1/2$, and the third quartile is obtained by taking $y = 3/4$.

The qf can also be used to generate random samples from the resulting asymmetric continuous distribution. The key result is that for a random variable Y with uniform distribution over $(0,1)$, the random variable

$$X = Q(Y) = \ell^{-1}\left[\frac{a-F^{-1}\{Y[F(b)-F(a)]+F(a)\}}{F^{-1}\{Y[F(b)-F(a)]+F(a)\}-b}\right],$$

has the cdf G. Since we can easily generate values from Y, we immediately derive generated values for X by proceeding as a substitution.

Furthermore, the qf can be used to define robust measures of the skewness and kurtosis of the distribution. We think of the famous Bowley skewness and Moors kurtosis. See [25, 26].

In addition, a moment analysis can be performed. If it exists, for any integer r, the r-th moment associated with the proposed distribution is given by

$$\mu_r = \int_{-\infty}^{+\infty} x^r g(x)dx = \frac{b-a}{F(b)-F(a)}\int_{-\infty}^{+\infty} x^r \frac{\ell'(x)}{[1+\ell(x)]^2} f\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]dx,$$

or, equivalently, with the use of the qf,

$$\mu_r = \int_0^1 [Q(y)]^r dy = \int_0^1 \left\{\ell^{-1}\left[\frac{a-F^{-1}\{y[F(b)-F(a)]+F(a)\}}{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}\right]\right\}^r dy.$$

No closed form expressions are expected for these integrals; only a computational effort seems reasonable to determine them. On the basis of these moments, we can define traditional measures such as the mean, variance, dispersion index, moment skewness, moment kurtosis, and so on. However, since their existence is not always guaranteed, quantile analysis is recommended. Therefore, we will not develop this aspect further.

3. New asymmetric distributions

In this section, we present new asymmetric (continuous) distributions based on Proposition 2.1, two standard baseline distributions with support $\mathbb{R}$: the normal and Cauchy distributions, and the functions ℓ described in Lemma 2.2.

3.1. Asymmetric normal distributions

This part focuses on the normal distribution as the chosen baseline distribution in Proposition 2.1.

3.1.1 Type 1 asymmetric normal distribution

The normal distribution with parameter $\mu \in \mathbb{R}$ and $\sigma > 0$ is defined by the following cdf:

$$F(x) = \frac{1}{2}\left\{1 + \operatorname{erf}\left[\frac{x-\mu}{\sigma\sqrt{2}}\right]\right\}, \quad x \in \mathbb{R},$$

where erf is the error function given as $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)dt$. The corresponding pdf is given as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad x \in \mathbb{R}.$$

It is clearly symmetric around $x = \mu$, i.e., $f(\mu - x) = f(\mu + x)$ for any $x \in \mathbb{R}$. More precisely, the curve of this pdf is symmetric and bell-shaped, centered at μ and with a standard deviation of σ.

The last function of particular interest is the corresponding qf, given by

$$F^{-1}(y) = \mu + \sigma\sqrt{2}\operatorname{erf}^{-1}(2y-1), \quad y \in (0,1),$$

where erf^{-1} is the inverse error function. Note that the normal distribution is implemented in most mathematical software. This makes it easy to manipulate the associated functions.

Apart from the functional considerations, the normal distribution is widely used in statistics because of the central limit theorem, which states that the sum of many independent random variables tends to follow a normal distribution, regardless of their original distributions. Of course, it is not suitable for the analysis of asymmetric data, which justifies the development of different types of asymmetric normal distributions, as motivated in the introduction of the article.

Based on Proposition 2.1, we introduce the type 1 asymmetric normal (T1AN) distribution with the normal distribution as the baseline distribution, and the function $\ell(x) = \exp(x)$. Thus, it is defined with the following cdf:

$$G(x) = \frac{1}{F(b) - F(a)}\left\{F\left[\frac{a + b\exp(x)}{1 + \exp(x)}\right] - F(a)\right\}, \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b) - F(a)} \frac{\exp(x)}{[1 + \exp(x)]^2} f\left[\frac{a + b\exp(x)}{1 + \exp(x)}\right], \quad x \in \mathbb{R}.$$

To demonstrate the asymmetry behavior of the T1AN distribution and all future distributions based on Proposition 2.1, we perform a graphical analysis. Figure 1 shows the curve of this pdf in the standard case, i.e., $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

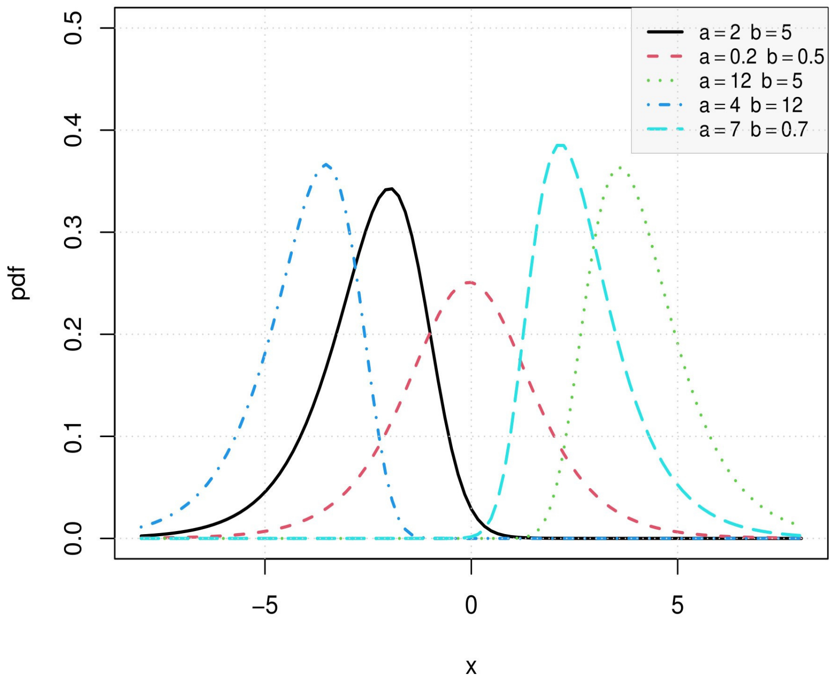

Figure 1. Curves of the pdf of the T1AN distribution for $x \in (-8, 8)$, $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

To highlight the fact that a and b can be negative, with $a < b$ or $b > a$, Figure 2 shows the curve of this pdf still for $\mu = 0$ and $\sigma = 1$, but for possible negative values of a and b.

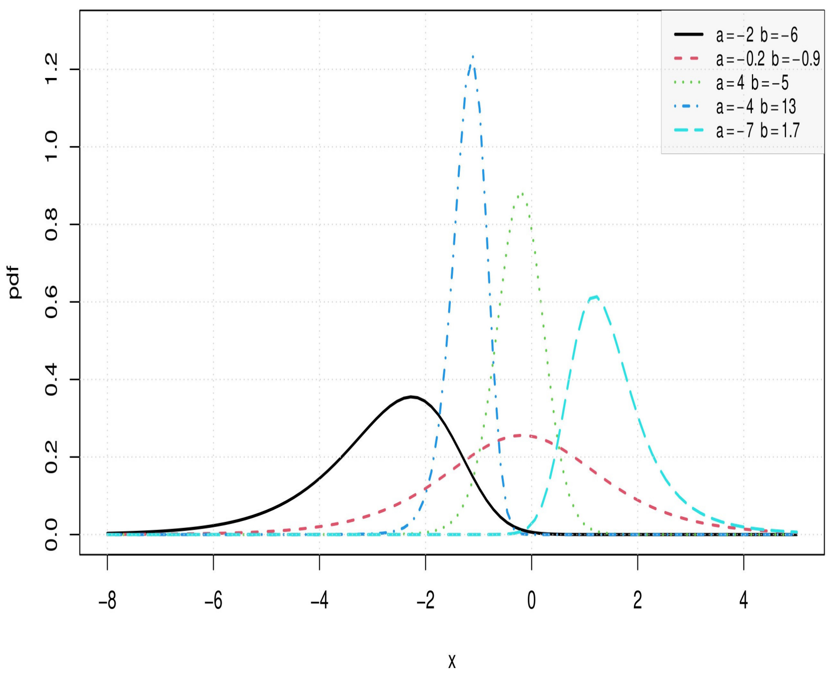

Figure 2. Curves of the pdf of the T1AN distribution for $x \in (-8, 5)$, $\mu = 0$ and $\sigma = 1$, and positive and negative values for a and b.

In these figures, we see that a and b affect the standard bell shape of the pdf of the standard normal distribution. A wide variety of asymmetric shapes are observed, from left to skewed, with varying degrees of flatness and weights on the tails. There are notable effects on location and deviation. We also mention that the activation of μ and σ can enhance these aspects; we have set $\mu = 0$ and $\sigma = 1$ to simplify the situation and to study the role of a and b. This graphical study confirms the interest we can put on the T1AN distribution as an asymmetric model from a statistical point of view.

To conclude this part, Proposition 2.3 gives the qf of the T1AN distribution. Since $\ell^{-1}(x) = \ln(x)$ with $x \in (0, +\infty)$, it can be expressed as

$$Q(y) = \ln\left[\frac{a - F^{-1}\{y[F(b) - F(a)] + F(a)\}}{F^{-1}\{y[F(b) - F(a)] + F(a)\} - b}\right], \quad y \in (0, 1).$$

Figure 3 shows the curve of this qf for $\mu = 0$ and $\sigma = 1$, and selected values for a and b, just to check its validity.

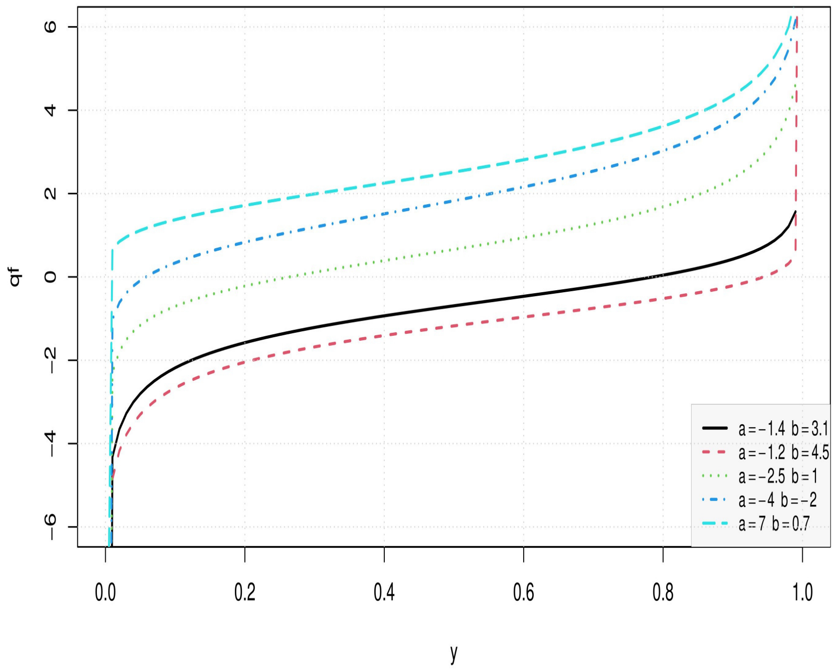

Figure 3. Curves of the qf of the T1AN distribution for $y \in (0, 1)$, $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

We observe the curve of a typical qf of a distribution with support $\mathbb{R}$. In particular, we clearly see $\lim_{y \to 0} Q(y) = -\infty$ and $\lim_{y \to 1} Q(y) = +\infty$.

3.1.2 Type 2 asymmetric normal distribution

Based on Proposition 2.1, we describe the type 2 asymmetric normal (T2AN) distribution with the normal distribution as the baseline distribution, and the function $\ell(x) = x + \sqrt{x^2 + 1}$. Thus, compared to the T1AN distribution, we consider a polynomial-power function for ℓ instead of the exponential function, which is known to have a slower rate of convergence (or divergence). The T2AN distribution is thus defined with the following cdf:

$$G(x) = \frac{1}{F(b) - F(a)}\left[F\left\{\frac{a + b[x + \sqrt{x^2 + 1}]}{1 + x + \sqrt{x^2 + 1}}\right\} - F(a)\right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b - a}{F(b) - F(a)} \frac{1}{[1 + x + \sqrt{x^2 + 1}]^2}\left[1 + \frac{x}{\sqrt{x^2 + 1}}\right] f\left\{\frac{a + b[x + \sqrt{x^2 + 1}]}{1 + x + \sqrt{x^2 + 1}}\right\}, \quad x \in \mathbb{R}.$$

Figure 4 shows the curve of this pdf for $\mu = 0$ and $\sigma = 1$, and positive values for a and b, to analyze the asymmetric effect.

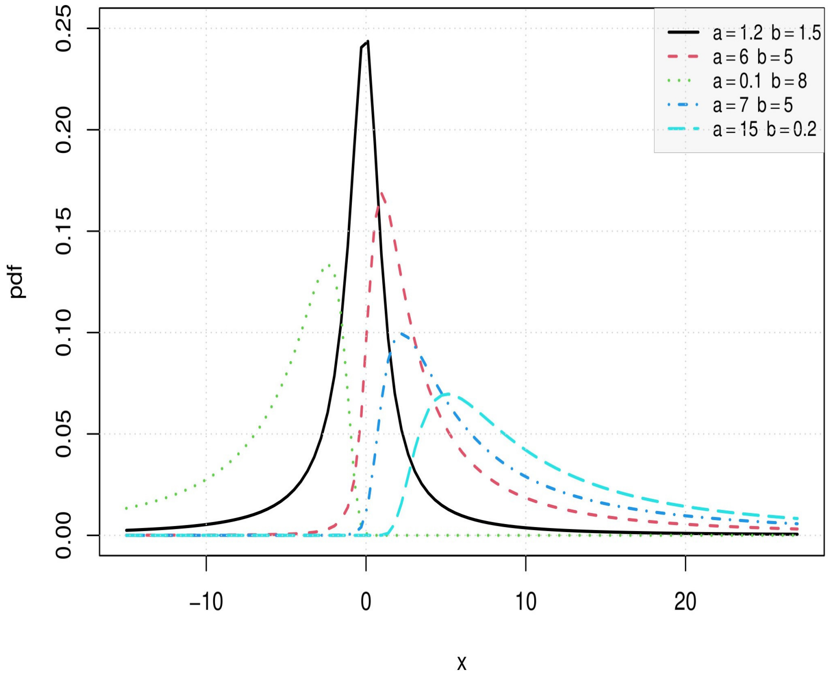

Figure 4. Curves of the pdf of the T2AN distribution for $x \in (-15, 27)$, $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

Figure 5 completes this by showing the curve of this pdf still for $\mu = 0$ and $\sigma = 1$, but for possible negative values of a and b.

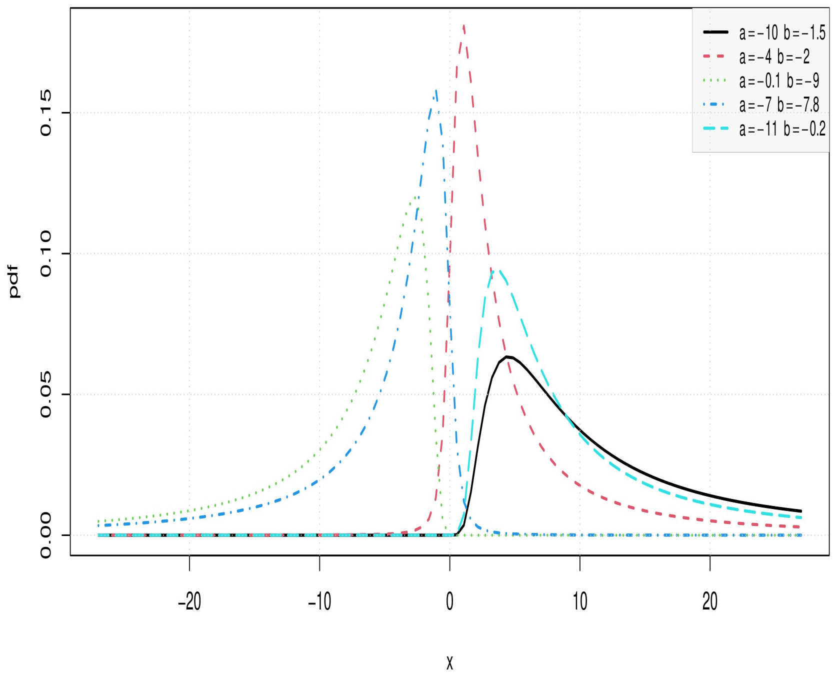

Figure 5. Curves of the pdf of the T2AN distribution for $x \in (-27, 27)$, $\mu = 0$ and $\sigma = 1$, and positive and negative values for a and b.

From these figures, we can see that the spike driven by the mode can be very pronounced and that the bell shape of the curve of the pdf of the standard normal distribution can be drastically skewed to the left or to the right. The asymmetry of the proposed strategy can be truly extreme in this sense.

To end this part, Proposition 2.3 gives the qf of the T2AN distribution. Using

$$\ell^{-1}(x) = \frac{x^2-1}{2x}, \quad x \in (0,+\infty),$$

it is expressed as

$$Q(y) = \frac{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}{2\left[a-F^{-1}\{y[F(b)-F(a)]+F(a)\}\right]}\left\{\left[\frac{a-F^{-1}\{y[F(b)-F(a)]+F(a)\}}{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}\right]^2-1\right\},$$
$$y \in (0,1).$$

This can be considered for further quantile analysis.

3.1.3 Type 3 asymmetric normal distribution

Based on Proposition 2.1, we define the type 3 asymmetric normal (T3AN) distribution with the normal distribution as the baseline distribution, and the function $\ell(x) = \exp[x+\sin(x)]$. The main originality of this choice is the presence of the sine function that conceptually injects oscillatory shapes into the main functions. Thus, the T3AN distribution is defined with the following cdf:

$$G(x) = \frac{1}{F(b)-F(a)}\left[F\left\{\frac{a+b\exp[x+\sin(x)]}{1+\exp[x+\sin(x)]}\right\}-F(a)\right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b)-F(a)}\frac{[1+\cos(x)]\exp[x+\sin(x)]}{[1+\exp[x+\sin(x)]]^2}f\left\{\frac{a+b\exp[x+\sin(x)]}{1+\exp[x+\sin(x)]}\right\}, \quad x \in \mathbb{R}.$$

Figure 6 presents the curve of this pdf for $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

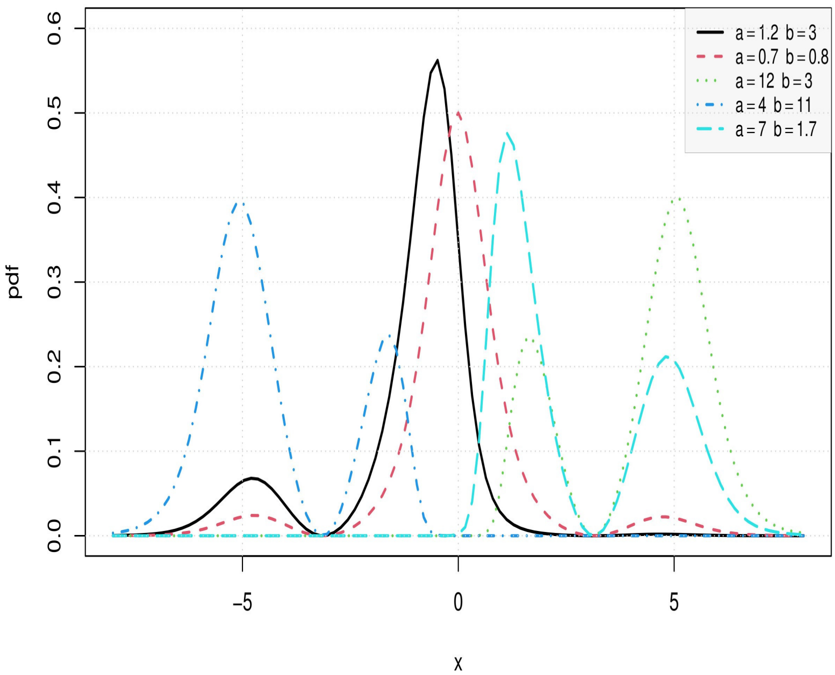

Figure 6. Curves of the pdf of the T3AN distribution for $x \in (-8,8)$, $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

To complete this graphical analysis, Figure 7 shows the curve of this pdf still for $\mu = 0$ and $\sigma = 1$, but for possible negative values of a and b.

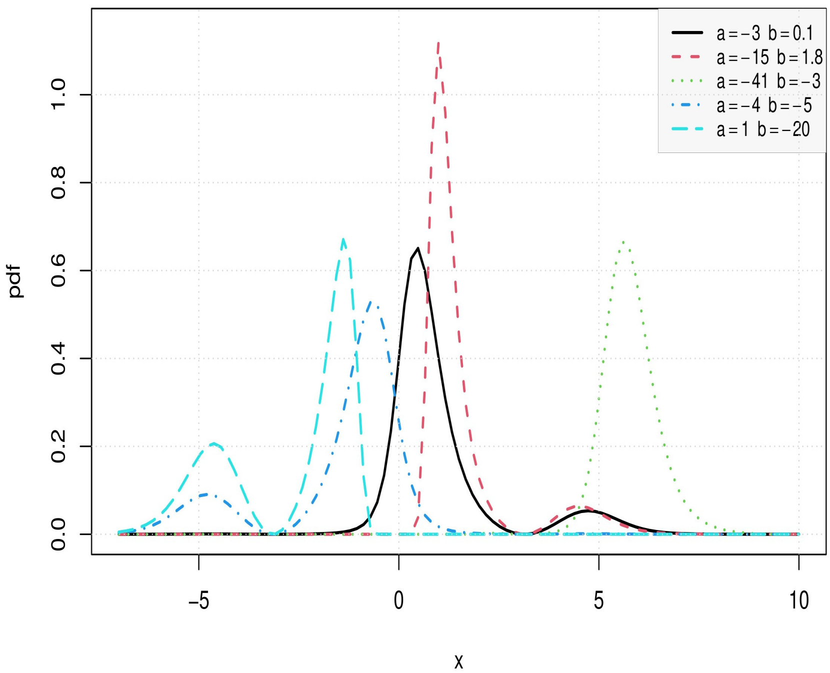

Figure 7. Curves of the pdf of the T3AN distribution for $x \in (-7, 10)$, $\mu = 0$ and $\sigma = 1$, and positive and negative values for a and b.

From these figures, we can see that the T3AN distribution can produce bimodal and trimodal asymmetric pdfs. This phenomenon is explained by the complex action of the sine function. Different levels of flatness are also observed. Therefore, the T3AN distribution can be an interesting option to analyze data whose histogram shows skewness and multimodality.

3.1.4 Type 4 asymmetric normal distribution

Based on Proposition 2.1, we introduce the type 4 asymmetric normal (T4AN) distribution with the normal distribution as the baseline distribution, and the function $\ell(x) = \ln[\exp(x) + 1]$. We can see that when $x \to +\infty$ we have $\ell(x) \sim x$ and when $x \to -\infty$ we have $\ell(x) \sim \exp(x)$. So there are two different rates of divergence at the extremes. The T4AN distribution is defined by the following cdf:

$$G(x) = \frac{1}{F(b) - F(a)} \left[F\left\{ \frac{a + b\ln[\exp(x) + 1]}{1 + \ln[\exp(x) + 1]} \right\} - F(a) \right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b - a}{F(b) - F(a)} \frac{\exp(x)}{[\exp(x) + 1]\{1 + \ln[\exp(x) + 1]\}^2} f\left\{ \frac{a + b\ln[\exp(x) + 1]}{1 + \ln[\exp(x) + 1]} \right\}, \quad x \in \mathbb{R}.$$

In Figure 8, for $\mu = 0$ and $\sigma = 1$, and positive values for a and b, we illustrate the asymmetric behavior of this pdf.

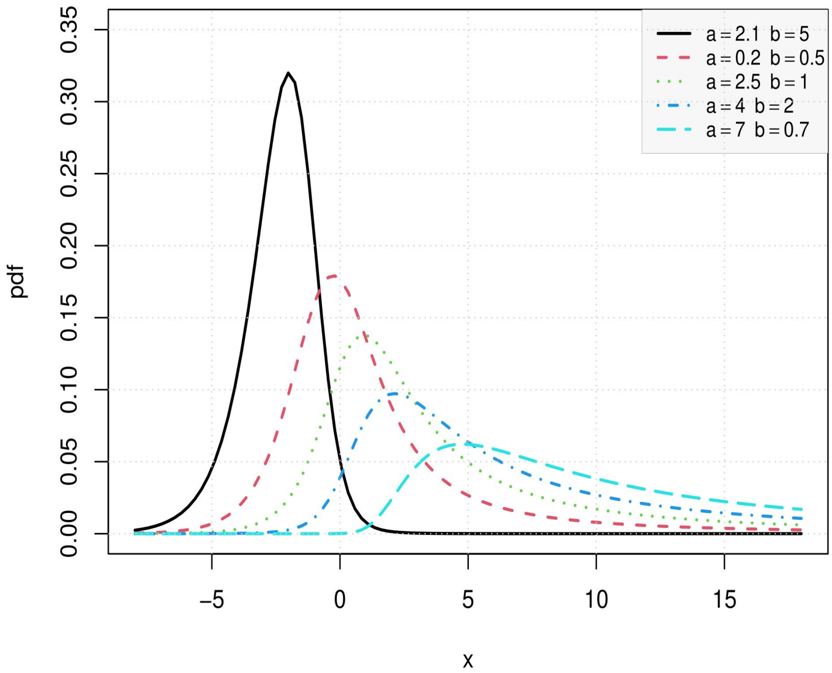

Figure 8. Curves of the pdf of the T4AN distribution for $x \in (-8, 18)$, $\mu = 0$ and $\sigma = 1$, and positive values for a and b.

Figure 9 completes this by showing the curve of this pdf still for $\mu = 0$ and $\sigma = 1$, but for possible negative values of a and b.

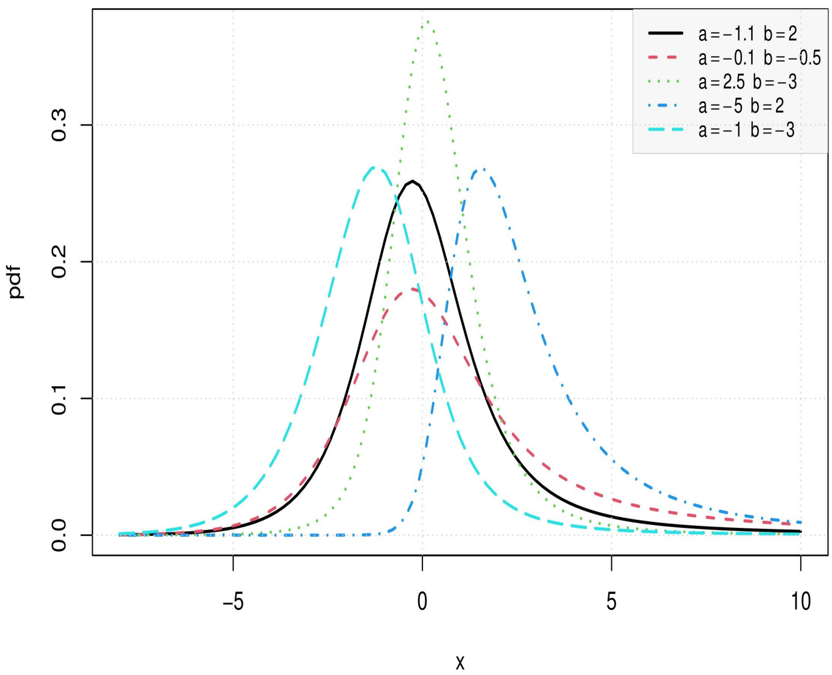

Figure 9. Curves of the pdf of the T4AN distribution for $x \in (-8, 10)$, $\mu = 0$ and $\sigma = 1$, and positive and negative values for a and b.

From these figures, we can see how the proposed strategy skews the bell shape of the curve of the pdf of the standard normal distribution, depending on the combined effect of a and b. The different levels of asymmetry depend in an obvious way on the values of a and b. The T4AN distribution appears to have a more pronounced right-skewness property than the T1AN distribution. It can be considered as an alternative in this sense.

To conclude this part, Proposition 2.3 gives the qf of the T4AN distribution. Using $\ell^{-1}(x) = \ln[\exp(x) - 1]$ with $x \in (0, +\infty)$, it is expressed as

$$Q(y) = \ln\left\{\exp\left[\frac{a - F^{-1}\{y[F(b) - F(a)] + F(a)\}}{F^{-1}\{y[F(b) - F(a)] + F(a)\} - b}\right] - 1\right\}, \quad y \in (0,1).$$

This function is the starting point for a deeper quantile analysis of the T4AN distribution.

In the next subsection, the previous asymmetric distributions are revisited, but with the consideration of a different baseline distribution than the normal distribution.

3.2. Asymmetric Cauchy distributions

This part focuses on the Cauchy distribution as the baseline distribution chosen in Proposition 2.1. We still consider the functions described in Lemma 2.2.

3.2.1 Type 1 asymmetric Cauchy distribution

The Cauchy distribution with parameter $x_0 \in \mathbb{R}$ and $\gamma > 0$ is defined by the following cdf:

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x - x_0}{\gamma}\right), \quad x \in \mathbb{R}.$$

The corresponding pdf is given as

$$f(x) = \frac{\gamma}{\pi[(x - x_0)^2 + \gamma^2]}, \quad x \in \mathbb{R}.$$

This pdf is thus of polynomial decay, contrary to that of the normal distribution. It is clearly symmetric around $x = x_0$, i.e., $f(x_0 - x) = f(x_0 + x)$ for any $x \in \mathbb{R}$. In addition, the corresponding qf is given as

$$F^{-1}(y) = x_0 + \gamma\tan\left[\pi\left(y - \frac{1}{2}\right)\right], \quad y \in (0,1).$$

Note that the Cauchy distribution is well known. In particular, it is implemented in most mathematical software. This makes it easy to manipulate the associated functions.

Apart from the probability functions, the Cauchy distribution differs from the normal distribution in that it has heavier tails, resulting in undefined mean and variance (whereas the normal distribution has lighter tails with well-defined mean and variance). This difference means that the Cauchy distribution is more likely to produce extreme values than the normal distribution. It is obviously unsuitable for the analysis of asymmetric data, which is why asymmetric versions of the Cauchy distribution have been developed, as discussed in the introduction to this article.

Based on Proposition 2.1, we introduce the type 1 asymmetric Cauchy (T1AC) distribution with the Cauchy distribution as the baseline distribution, and the function $\ell(x) = \exp(x)$. Thus, it is defined with the following cdf:

$$G(x) = \frac{1}{F(b) - F(a)}\left\{F\left[\frac{a + b\exp(x)}{1 + \exp(x)}\right] - F(a)\right\}, \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b - a}{F(b) - F(a)}\frac{\exp(x)}{[1 + \exp(x)]^2}f\left[\frac{a + b\exp(x)}{1 + \exp(x)}\right], \quad x \in \mathbb{R}.$$

Let us now perform a graphical analysis of this pdf. Figure 10 shows the curve of this pdf for $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

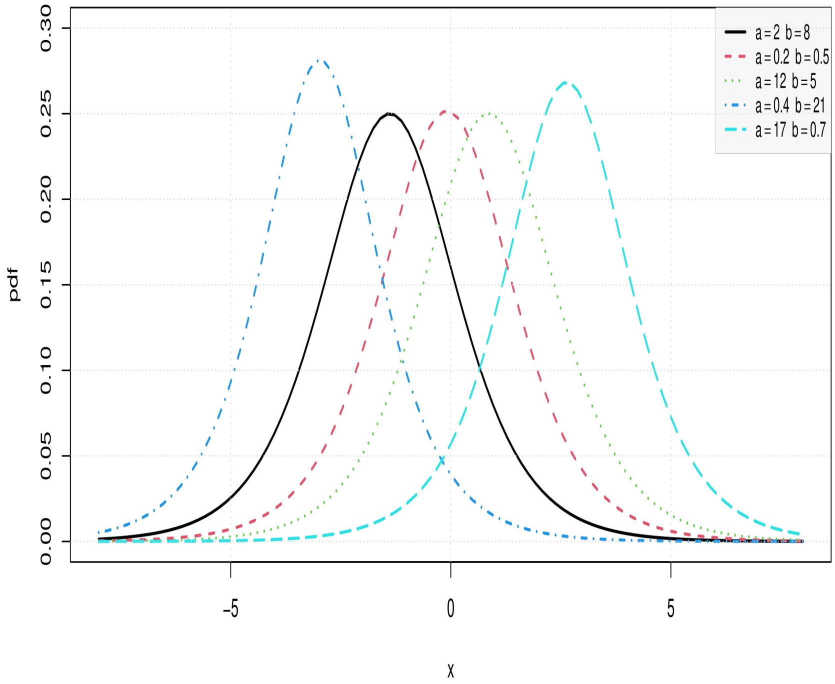

Figure 10. Curves of the pdf of the T1AC distribution for $x \in (-8, 8)$, $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

To highlight the fact that a and b can be negative, with $a < b$ or $b > a$, Figure 11 shows the curve of this pdf still for $x_0 = 0$ and $\gamma = 1$, but for possible negative values of a and b.

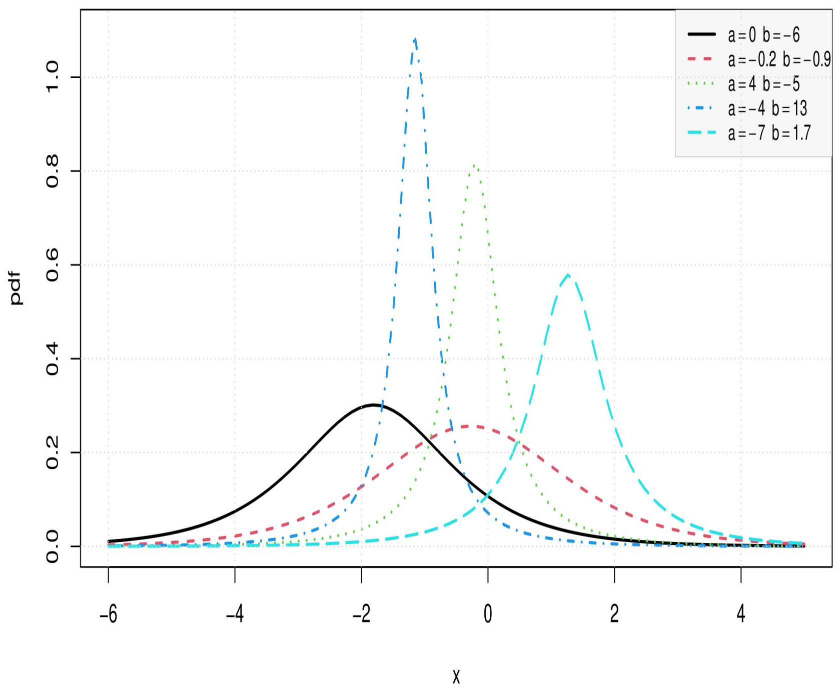

Figure 11. Curves of the pdf of the T1AC distribution for $x \in (-6, 5)$, $x_0 = 0$ and $\gamma = 1$, and positive and negative values for a and b.

In these figures, we see a wide variety of non-abrupt asymmetric shapes, from left to right, with varying degrees of flatness. The appearance is close to that of Figure 1. In this sense, given the characteristics of the Cauchy distribution, it can be considered as an alternative to the T1AN distribution. This graphical study demonstrates the statistical interest of the T1AC distribution as an asymmetric model.

Proposition 2.3 gives the qf of the T1AC distribution. Since $\ell^{-1}(x) = \ln(x)$ with $x \in (0, +\infty)$, it can be expressed as

$$Q(y) = \ln\left[\frac{a - F^{-1}\{y[F(b) - F(a)] + F(a)\}}{F^{-1}\{y[F(b) - F(a)] + F(a)\} - b}\right], \quad y \in (0,1).$$

A quantile analysis is thus quite possible. Figure 12 presents the curve of this qf for $x_0 = 0$ and $\gamma = 1$, and selected values for a and b.

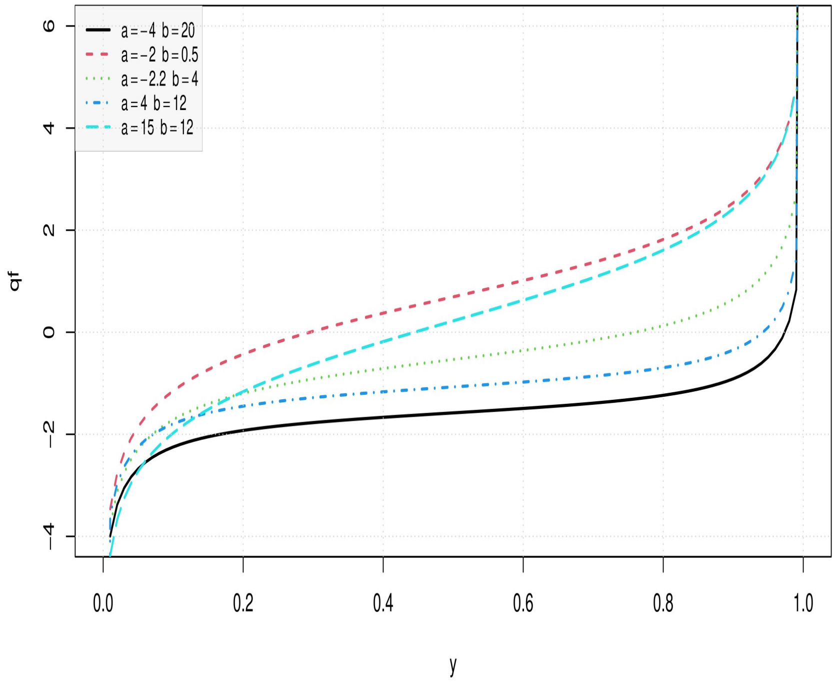

Figure 12. Curves of the qf of the T1AC distribution for $y \in (0,1)$, $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

The properties of a typical qf of a distribution with support $\mathbb{R}$ are observed.

We end this part with a remark on the moments of the T1AC distribution: by choosing $\ell(x) = \exp(x)$ and by analyzing the integrability of the corresponding pdf, the T1AC distribution admits moments of any order, unlike the Cauchy distribution. The proposed strategy is thus able to allow moment analysis, even though the baseline distribution is limited in this respect.

3.2.2 Type 2 asymmetric Cauchy distribution

Based on Proposition 2.1, we present the type 2 asymmetric Cauchy (T2AC) distribution with the Cauchy distribution as the baseline distribution, and the function $\ell(x) = x + \sqrt{x^2 + 1}$. Thus, it is defined with the following cdf:

$$G(x) = \frac{1}{F(b) - F(a)}\left[F\left\{\frac{a + b[x + \sqrt{x^2+1}]}{1 + x + \sqrt{x^2+1}}\right\} - F(a)\right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b) - F(a)} \frac{1}{[1 + x + \sqrt{x^2+1}]^2}\left[1 + \frac{x}{\sqrt{x^2+1}}\right] f\left\{\frac{a + b[x + \sqrt{x^2+1}]}{1 + x + \sqrt{x^2+1}}\right\}, \quad x \in \mathbb{R}.$$

Taking into account the polynomial nature of f, the pdf g is entirely of polynomial nature. In this respect, the difference with the pdf of the T2AN distribution is notable. Figure 13 shows the curve of this pdf for $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

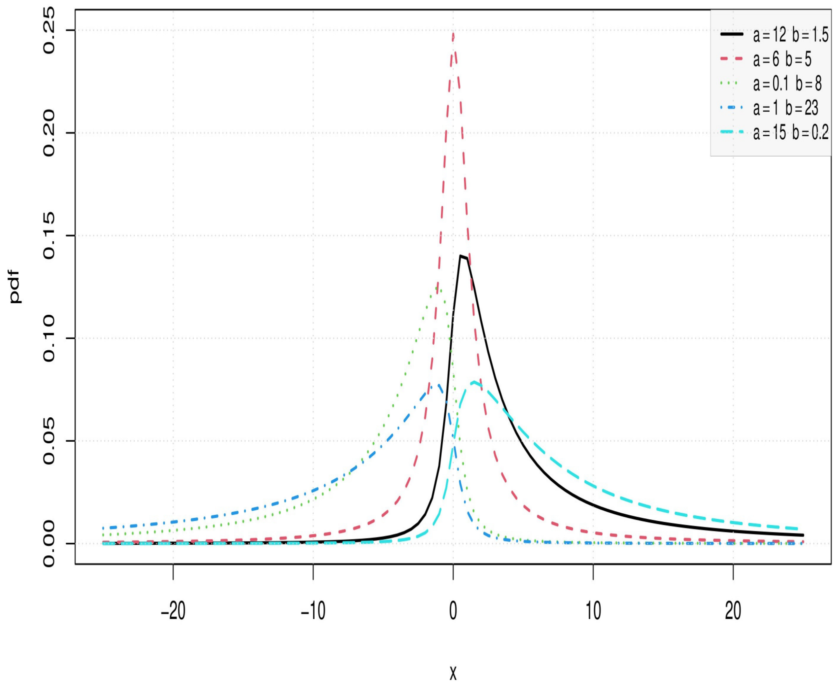

Figure 13. Curves of the pdf of the T2AC distribution for $x \in (-25, 25)$, $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

Figure 14 completes this by showing the curve of this pdf still for $x_0 = 0$ and $\gamma = 1$, but for possible negative values of a and b.

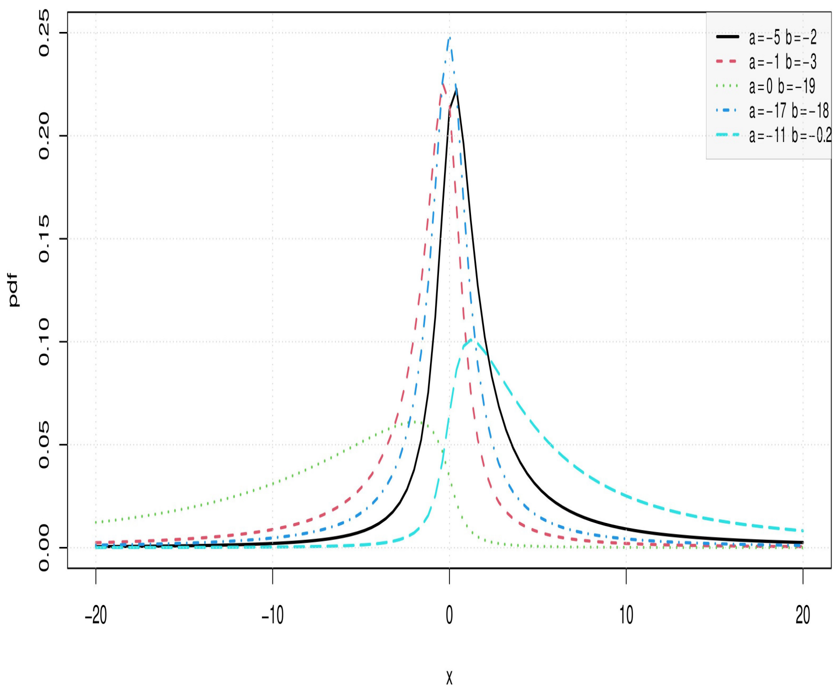

Figure 14. Curves of the pdf of the T2AC distribution for $x \in (-20, 20)$, $x_0 = 0$ and $\gamma = 1$, and positive and negative values for a and b.

The most remarkable points in these figures are the peaks of the curves, which are very pronounced, and the different levels of skewness to the left or to the right. The sharpness of the peak is clearly the main difference between the pdfs of the T1AC and T2AC distributions.

To end this part, Proposition 2.3 gives the qf of the T2AC distribution. Using

$$\ell^{-1}(x) = \frac{x^2-1}{2x}, \quad x \in (0,+\infty),$$

it is expressed as

$$Q(y) = \frac{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}{2\left[a-F^{-1}\{y[F(b)-F(a)]+F(a)\}\right]}\left\{\left[\frac{a-F^{-1}\{y[F(b)-F(a)]+F(a)\}}{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}\right]^2-1\right\},$$
$$y \in (0,1).$$

It can be used in the same way as it is for quantile analysis.

3.2.3 Type 3 asymmetric Cauchy distribution

Based on Proposition 2.1, we introduce the type 3 asymmetric Cauchy (T3AC) distribution with the Cauchy distribution as the baseline distribution, and the function $\ell(x) = \exp[x+\sin(x)]$. We thus follow the same scheme as for the T3AN distribution, with the aim of exploiting the oscillatory nature of the sine function to produce multimodality. The T3AC distribution is defined by the following cdf:

$$G(x) = \frac{1}{F(b)-F(a)}\left[F\left\{\frac{a+b\exp[x+\sin(x)]}{1+\exp[x+\sin(x)]}\right\}-F(a)\right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b)-F(a)}\frac{[1+\cos(x)]\exp[x+\sin(x)]}{[1+\exp[x+\sin(x)]]^2}f\left\{\frac{a+b\exp[x+\sin(x)]}{1+\exp[x+\sin(x)]}\right\}, \quad x \in \mathbb{R}.$$

Figure 15 presents the curve of this pdf for $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

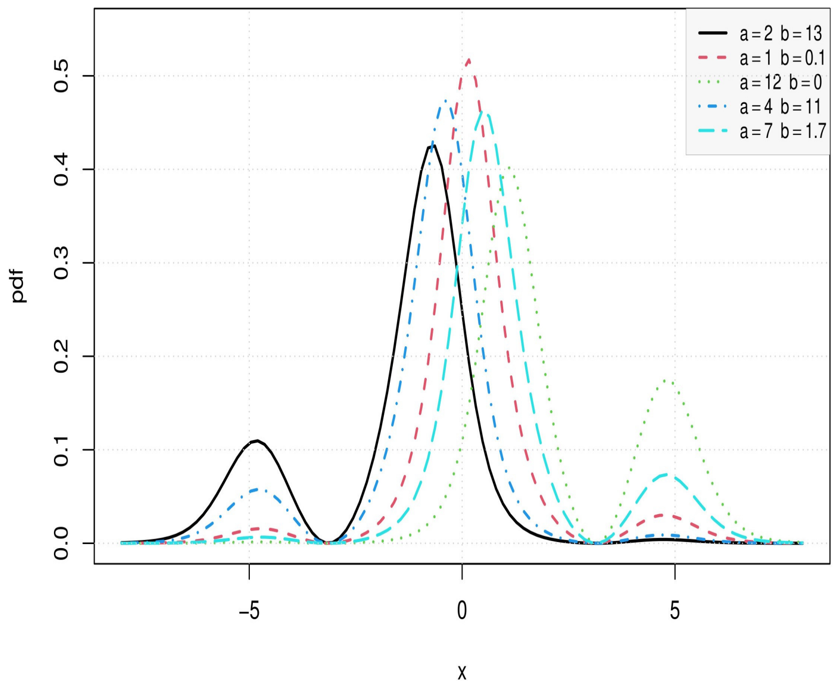

Figure 15. Curves of the pdf of the T3AC distribution for $x \in (-8,8)$, $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

This is completed in Figure 16 by showing the curve of this pdf still for $x_0 = 0$ and $\gamma = 1$, but for possible negative values of a and b.

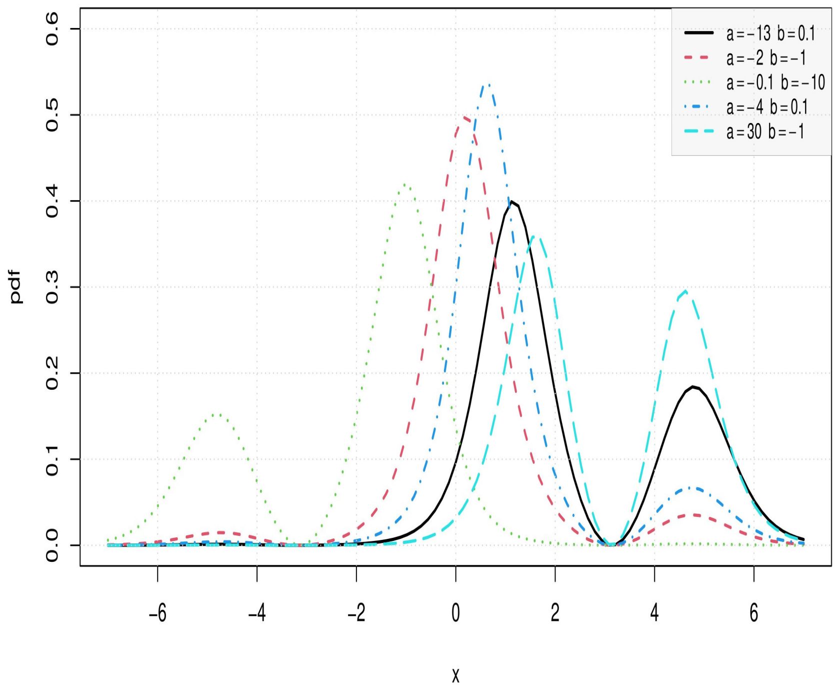

Figure 16. Curves of the pdf of the T3AC distribution for $x \in (-7,7)$, $x_0 = 0$ and $\gamma = 1$, and positive and negative values for a and b.

These figures show one, two or three modes, sometimes significant. This, combined with the various asymmetric levels, makes the T3AC distribution attractive for data with a histogram having these characteristics.

3.2.4 Type 4 asymmetric Cauchy distribution

Based on Proposition 2.1, we present the type 4 asymmetric Cauchy (T4AC) distribution with the Cauchy distribution as the baseline distribution, and the function $\ell(x) = \ln[\exp(x)+1]$. Thus, it is defined with the following cdf:

$$G(x) = \frac{1}{F(b)-F(a)}\left[F\left\{\frac{a+b\ln[\exp(x)+1]}{1+\ln[\exp(x)+1]}\right\} - F(a)\right], \quad x \in \mathbb{R},$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b)-F(a)} \frac{\exp(x)}{[\exp(x)+1]\{1+\ln[\exp(x)+1]\}^2} f\left\{\frac{a+b\ln[\exp(x)+1]}{1+\ln[\exp(x)+1]}\right\}, \quad x \in \mathbb{R}.$$

Thus, it provides a polynomial-power alternative to the pdf of the T4AN distribution. Figure 17 presents the curve of this pdf for $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

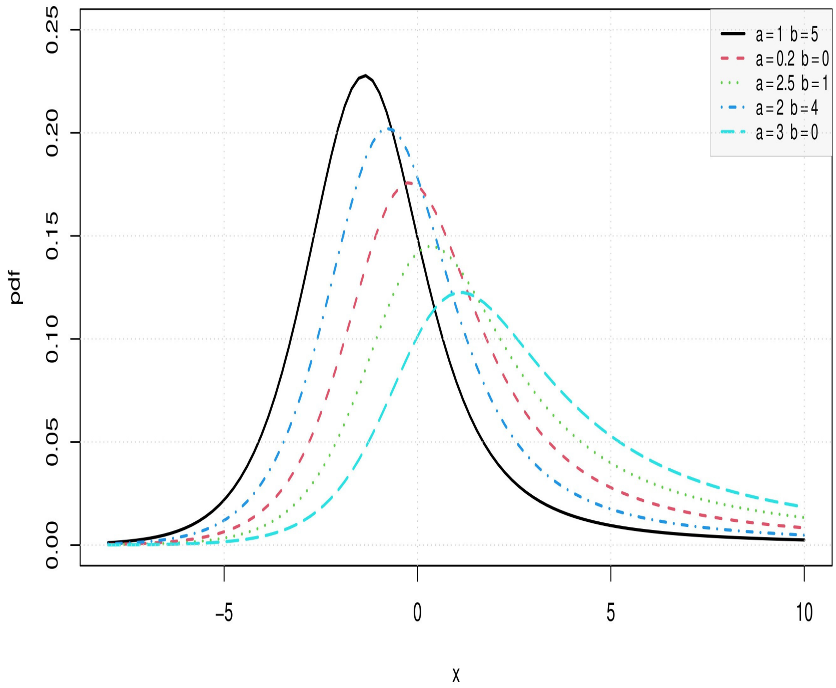

Figure 17. Curves of the pdf of the T4AC distribution for $x \in (-8, 10)$, $x_0 = 0$ and $\gamma = 1$, and positive values for a and b.

Figure 18 completes this by showing the curve of this pdf still for $x_0 = 0$ and $\gamma = 1$, but for possible negative values of a and b.

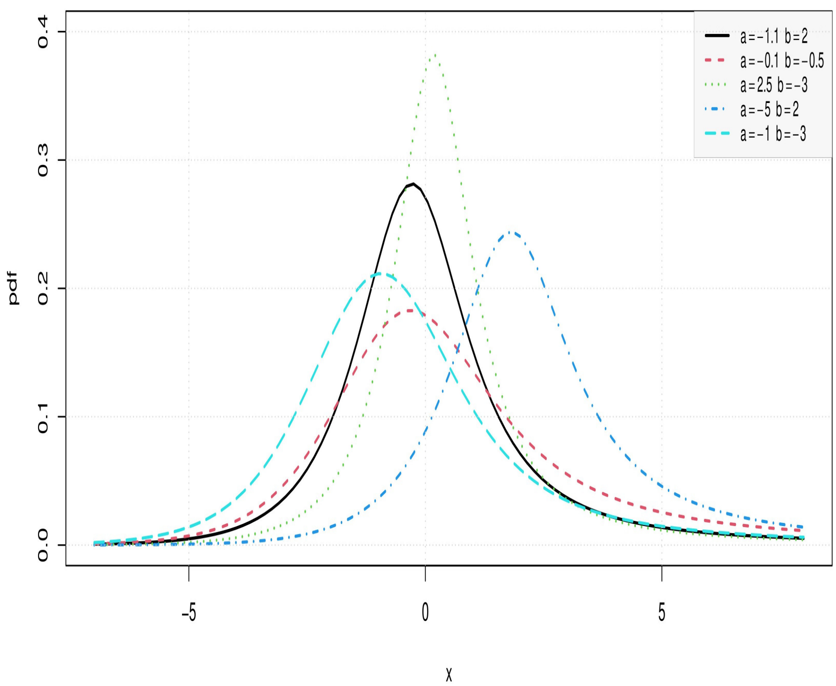

Figure 18. Curves of the pdf of the T4AC distribution for $x \in (-7, 8)$, $x_0 = 0$ and $\gamma = 1$, and positive and negative values for a and b.

From these figures, we can see the asymmetric behavior of the proposed strategy; the bell shape of the curve of the pdf of the standard Cauchy distribution can be drastically skewed from left to right.

To end this part, Proposition 2.3 gives the qf of the T4AC distribution. Using $\ell^{-1}(x) =$

$\ln[\exp(x)-1]$ with $x \in (0,+\infty)$, it is expressed as

$$Q(y) = \ln\left\{\exp\left[\frac{a-F^{-1}\{y[F(b)-F(a)]+F(a)\}}{F^{-1}\{y[F(b)-F(a)]+F(a)\}-b}\right]-1\right\}, \quad y \in (0,1).$$

This formula, which is quite manageable from a computational point of view, can be integrated into various quantile manipulations (calculation of quartiles, quantile regression models, etc.).

4. Complements

In this section, we provide some additional discussion and results concerning the possible baseline distributions and the adaptation of the asymmetric strategy for generating distributions with support $(0,+\infty)$.

4.1. On the baseline distribution

In this article, we have considered the normal and Cauchy distributions as baseline symmetric distributions with support $\mathbb{R}$ into Proposition 2.1. Of course, other basic distributions with this support can be studied, including those described below.

1. The logistic distribution with parameters $\mu \in \mathbb{R}$ and $s > 0$ defined by the following cdf, pdf and qf, respectively:

$$F(x) = \frac{1}{1+\exp[-(x-\mu)/s]}, \quad x \in \mathbb{R},$$

$$f(x) = \frac{\exp[-(x-\mu)/s]}{s\{1+\exp[-(x-\mu)/s]\}^2}, \quad x \in \mathbb{R},$$

and

$$Q(y) = \mu + s\ln\left(\frac{y}{1-y}\right), \quad y \in (0,1).$$

The logistic distribution is characterized by its sigmoid cdf. It is often used for modelling growth and logistic regression. In fact, it is similar to the normal distribution, but with heavier tails.

2. The Laplace distribution with parameters $\mu \in \mathbb{R}$ and $\beta > 0$ is defined by the following cdf, pdf and qf, respectively:

$$F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}(x-\mu)\left\{1-\exp\left[-\frac{|x-\mu|}{\beta}\right]\right\}, \quad x \in \mathbb{R},$$

where sign denotes the sign function defined as $\operatorname{sign}(x) = -1$ if $x < 0$, $\operatorname{sign}(x) = 0$ if $x = 0$ and $\operatorname{sign}(x) = 1$ if $x > 0$,

$$f(x) = \frac{1}{\beta}\exp\left[-\frac{|x-\mu|}{\beta}\right], \quad x \in \mathbb{R},$$

and

$$Q(y) = \mu - \beta\operatorname{sign}\left(y-\frac{1}{2}\right)\ln\left[1-2\left|y-\frac{1}{2}\right|\right], \quad y \in (0,1).$$

The Laplace distribution is characterized by its symmetric, peaked pdf at the mean and exponential tails. This makes it useful for modelling data with heavier tails than the normal distribution.

3. The Gumbel distribution with parameters $\mu \in \mathbb{R}$ and $\beta > 0$ is defined by the following cdf, pdf and qf, respectively:

$$F(x) = \exp\{-\exp[-(x-\mu)/\beta]\}, \quad x \in \mathbb{R},$$

$$f(x) = \frac{1}{\beta}\exp\{-(x-\mu)/\beta - \exp[-(x-\mu)/\beta]\}, \quad x \in \mathbb{R},$$

and

$$Q(y) = \mu - \beta \ln\{-\ln(y)\}, \quad y \in (0,1).$$

The Gumbel distribution is mainly used to model the distribution of the maximum (or minimum) of a number of samples from a given distribution.

4. The hyperbolic secant distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ is defined by the following cdf, pdf and qf, respectively:

$$F(x) = \frac{2}{\pi}\arctan\left\{\exp\left[\frac{\pi(x-\mu)}{2\sigma}\right]\right\}, \quad x \in \mathbb{R},$$

$$f(x) = \frac{1}{2\sigma}\operatorname{sech}\left[\frac{\pi(x-\mu)}{2\sigma}\right], \quad x \in \mathbb{R},$$

where sech denotes the hyperbolic secant function indicated as $\operatorname{sech}(x) = 2/[\exp(x) + \exp(-x)]$, and

$$Q(y) = \mu + \sigma\frac{2}{\pi}\ln\left[\tan\left(\frac{\pi y}{2}\right)\right], \quad y \in (0,1).$$

Taking $\mu = 0$ and $\sigma = 1$, the hyperbolic secant distribution shares several properties with the standard normal distribution. In particular, its pdf is proportional to its characteristic function. However, the hyperbolic secant distribution is leptokurtic, which means that it has a sharper peak near its mean and heavier tails than the standard normal distribution.

More details on these distributions can be found in [24].

Thus, by considering these symmetric distributions into Proposition 2.1, we can generate new asymmetric distributions, as we did in Section 3 for the normal and Cauchy distributions. This opens up an area of research in this direction.

4.2. With support on $(0,+\infty)$

Another interesting point is that Proposition 2.1 can be applied to non-necessarily symmetric distributions, and more generally to non-necessarily symmetric distributions with support $\mathbb{R}$. The result below is a modification of this proposition to deal with distributions with support $(0,+\infty)$.

Proposition 4.1 *Let us consider a continuous distribution with support on $(0,+\infty)$, cdf denoted by F and pdf denoted by f. Let $a \geq 0$ and $b \geq 0$ with $a \neq b$. Let ℓ be a differentiable increasing function on $(0,+\infty)$ such that $\lim_{x\to 0}\ell(x) = 0$ and $\lim_{x\to+\infty}\ell(x) = +\infty$. Let us now set*

$$G(x) = \frac{1}{F(b)-F(a)}\left\{F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right] - F(a)\right\}, \quad x > 0,$$

and $G(x) = 1$ for $x \leq 0$. Then G is a cdf that defines a new distribution with support on $(0,+\infty)$.

Note that we can have $a = 0$ so that $F(a) = 0$, or $b = 0$ so that $F(b) = 0$.

In particular, if the baseline distribution has monotonic shapes, the new distribution may transform it to introduce non-monotonicity and certain shape asymmetry, mainly depending on the definition of ℓ. To illustrate this claim, let us consider a basic setting. We choose the standard exponential distribution with support $(0,+\infty)$, i.e., with cdf $F(x) = 1-\exp(-x)$ for $x > 0$ and $F(x) = 0$ for $x \leq 0$, and pdf $f(x) = \exp(-x)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, and $\ell(x) = x+\sin(x)$, which satisfies the required conditions. Then, based on Proposition 4.1, the corresponding cdf is obtained as

$$G(x) = \frac{1}{F(b)-F(a)}\left[F\left\{\frac{a+b[x+\sin(x)]}{1+x+\sin(x)}\right\}-F(a)\right], \quad x > 0,$$

and $G(x) = 0$ for $x \leq 0$, with $a > 0$, $b > 0$, and $a \neq b$. The corresponding pdf is obtained as

$$g(x) = \frac{b-a}{F(b)-F(a)}\frac{1+\cos(x)}{[1+x+\sin(x)]^2}f\left\{\frac{a+b[x+\sin(x)]}{1+x+\sin(x)}\right\}, \quad x > 0,$$

and $g(x) = 0$ for $x \leq 0$. Figure 19 presents the curve of this pdf for selected positive values for a and b.

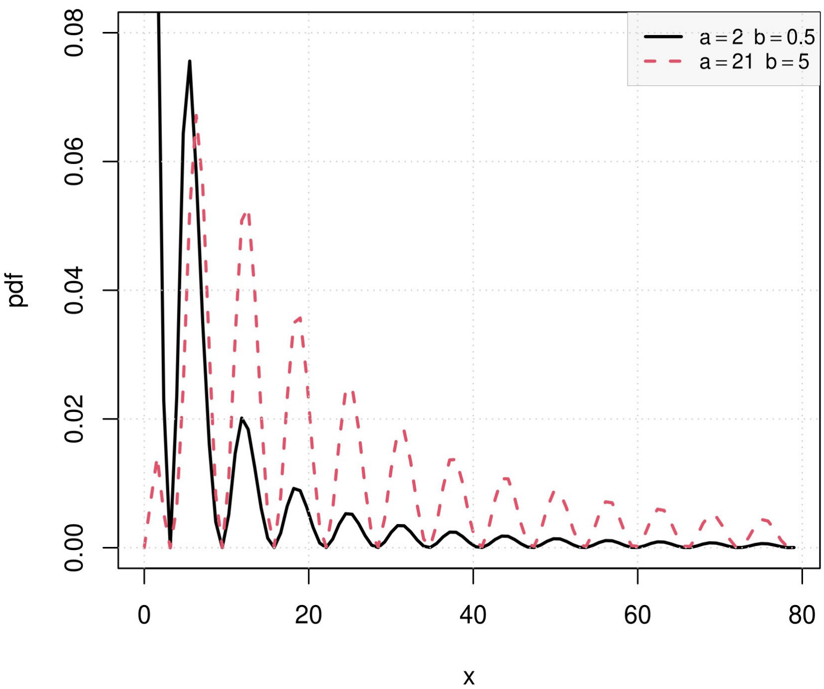

Figure 19. Curves of the pdf of the modified exponential distribution for $x \in (0,79)$, and positive values for a and b.

From this figure, we can see how the decrease of the pdf of the standard exponential distribution is significantly modified, mainly thanks to the asymmetry strategy, which includes a trigonometric function. It is thus adapted to the analysis of data with positive values and possible multiple modes, all of which have a decreasing pattern visible, for example, on the corresponding histogram. Our methodology can be used to construct various trigonometric lifetime distributions, in the spirit of [27].

We end this part by mentioning that the strategy can be adapted to distributions with bounded support, such as $(0,1)$, but this requires more constraints on a and b, making it excessively complicated. In a sense, we lose the advantage of having almost no assumptions about a and b. For this reason, we do not develop this aspect further.

5. Conclusion

Creating asymmetric continuous distributions is essential to model real-world phenomena where data often exhibit skewed behavior. Several options already exist. In order to innovate, it is necessary to develop a new thinking strategy with a solid theoretical foundation. In this article, we have made a contribution in this sense. We have proposed a new mathematical strategy designed to introduce asymmetry into any distribution with support of the entire real line. It depends on two tuning parameters that have almost no conditions on their value ranges, and an intermediate function that can modify the functionalities of the baseline structure. Some related theory has been demonstrated, with emphasis on quantile analysis. In addition, four types of asymmetric normal distributions and four types of asymmetric Cauchy distributions have been developed as applications of this setting. The importance of their asymmetric behavior was supported by an extensive graphical study. Some additional results revealed interesting facts in probability and statistics beyond the topic of asymmetric distributions.

The limitation of this work may lie in the deep interpretation of the role of the baseline distribution, the function ℓ and the parameters a and b on the overall asymmetry and shapes of the probability functions involved. Indeed, these elements are complexly interrelated, and it is difficult to isolate and analyse their shape effects. This explains why we have decided to propose a graphical approach, which seems much clearer than a mathematical development.

An interesting theoretical perspective is the extension of the strategy to the multidimensional setting. The practical perspective of this article is the analysis of data with different types of skewness. This analysis can be carried out using maximum the likelihood estimation method to estimate the parameters, which has been shown to be efficient in various application scenarios. This work requires extensive research into the parametric estimation aspect and the precise statistical analysis of modern data. We will leave this for another article.

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Conflicts of Interests

The author declares no conflict of interest.

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Appendix

This appendix contains the proofs of the main results.

Proof of Proposition 2.1 Since F and ℓ are continuous on $\mathbb{R}$ and $a \neq b$, G is continuous by the standard operations of continuous functions. Using the continuity of F and $\lim_{x\to-\infty}\ell(x) = 0$, we have

$$\begin{aligned}\lim_{x\to-\infty} G(x) &= \frac{1}{F(b)-F(a)}\left\{\lim_{x\to-\infty} F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}\left\{F\left[\lim_{x\to-\infty}\frac{a+b\ell(x)}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}\left\{F\left[\frac{a+b\times 0}{1+0}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}[F(a)-F(a)] = 0.\end{aligned}$$

Let us distinguish the case $b \neq 0$ and the case $b = 0$. For $b \neq 0$, by using the continuity of F and $\lim_{x\to+\infty}\ell(x) = +\infty$, we have

$$\begin{aligned}\lim_{x\to+\infty} G(x) &= \frac{1}{F(b)-F(a)}\left\{\lim_{x\to+\infty} F\left[\frac{a+b\ell(x)}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}\left\{F\left[\lim_{x\to+\infty}\frac{a+b\ell(x)}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}\left\{F\left[\lim_{x\to+\infty}\frac{b\ell(x)}{\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(b)-F(a)}[F(b)-F(a)] = 1.\end{aligned}$$

For $b = 0$, with the same arguments, we have

$$\begin{aligned}\lim_{x\to+\infty} G(x) &= \frac{1}{F(0)-F(a)}\left\{\lim_{x\to+\infty} F\left[\frac{a}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(0)-F(a)}\left\{F\left[\lim_{x\to+\infty}\frac{a}{1+\ell(x)}\right] - F(a)\right\}\\ &= \frac{1}{F(0)-F(a)}[F(0)-F(a)] = 1.\end{aligned}$$

Since $F' = f$, the differentiation of $G(x)$ with respect to x gives

$$G'(x) = \frac{1}{F(b)-F(a)}\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]' f\left[\frac{a+b\ell(x)}{1+\ell(x)}\right],$$

where

$$\begin{aligned}\left[\frac{a+b\ell(x)}{1+\ell(x)}\right]' &= \frac{b\ell'(x)[1+\ell(x)] - [a+b\ell(x)]\ell'(x)}{[1+\ell(x)]^2}\\ &= (b-a)\frac{\ell'(x)}{[1+\ell(x)]^2}.\end{aligned}$$

Hence, we have

$$G'(x) = \frac{b-a}{F(b)-F(a)}\frac{\ell'(x)}{[1+\ell(x)]^2} f\left[\frac{a+b\ell(x)}{1+\ell(x)}\right].$$

Since F is a cdf, it is an increasing function. Therefore, for any $a \neq b$, whatever the sign of a and b, we have
$$\frac{b-a}{F(b)-F(a)} > 0.$$
Furthermore, since ℓ is differentiable and increasing by assumptions, we have $\ell'(x) \geq 0$ for any $x \in \mathbb{R}$. Also, since f is a pdf, we always have $f(x) \geq 0$ for any $x \in \mathbb{R}$. So, as a product of positive functions, we have $G'(x) \geq 0$ for any $x \in \mathbb{R}$.

Combining these properties, we prove that G is a valid cdf. □

Proof of Lemma 2.2

1. By the properties of the exponential function, we have directly $\lim_{x\to-\infty} \ell(x) = \lim_{x\to-\infty} \exp(x) = 0$, $\lim_{x\to+\infty} \ell(x) = \lim_{x\to+\infty} \exp(x) = +\infty$, ℓ is differentiable and $\ell'(x) = \exp(x) \geq 0$, meaning that ℓ is increasing.
2. We have
$$\lim_{x\to-\infty} \ell(x) = \lim_{x\to-\infty} [x + \sqrt{x^2+1}] = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}-x} = 0,$$
$\lim_{x\to+\infty} \ell(x) = \lim_{x\to+\infty} [x + \sqrt{x^2+1}] = \lim_{x\to+\infty} 2x = +\infty$, ℓ is differentiable and
$$\ell'(x) = 1 + \frac{x}{\sqrt{x^2+1}}.$$
For any $x \geq 0$, it is clear that $\ell'(x) \geq 0$. For $x < 0$, we have $\sqrt{x^2+1} \geq \sqrt{x^2} = |x| = -x \geq 0$, implying that
$$\ell'(x) = 1 + \frac{x}{\sqrt{x^2+1}} \geq 1 - \frac{x}{x} = 0.$$
Hence, ℓ is increasing for any $x \in \mathbb{R}$.
3. Since $\lim_{x\to\pm\infty} \sin(x)/x = 0$, we have $\lim_{x\to-\infty} \ell(x) = \lim_{x\to-\infty} \exp[x + \sin(x)] = \lim_{x\to-\infty} \exp(x) = 0$, $\lim_{x\to+\infty} \ell(x) = \lim_{x\to+\infty} \exp[x+\sin(x)] = \lim_{x\to+\infty} \exp(x) = +\infty$, ℓ is differentiable and, since $\cos(x) \geq -1$,
$$\ell'(x) = [1+\cos(x)] \exp[x+\sin(x)] \geq 0,$$
meaning that ℓ is increasing.
4. We have $\lim_{x\to-\infty} \ell(x) = \lim_{x\to-\infty} \ln[\exp(x) + 1] = \ln(0+1) = 0$, $\lim_{x\to+\infty} \ell(x) = \lim_{x\to+\infty} \ln[\exp(x)+1] = \lim_{x\to+\infty} x = +\infty$, ℓ is differentiable and
$$\ell'(x) = \frac{\exp(x)}{\exp(x)+1} \geq 0,$$
meaning that ℓ is increasing.

This ends the proof. □

Proof of Proposition 2.3 The qf Q is defined by $Q(y) = G^{-1}(y)$ with $y \in (0,1)$. Based on this, by putting $x = Q(y)$, we have the following equivalences:

$$\begin{aligned}
G(x) = y \Leftrightarrow & \frac{1}{F(b) - F(a)} \left\{ F\left[\frac{a + b\ell(x)}{1 + \ell(x)}\right] - F(a) \right\} = y \\
\Leftrightarrow & F\left[\frac{a + b\ell(x)}{1 + \ell(x)}\right] = y[F(b) - F(a)] + F(a) \\
\Leftrightarrow & \frac{a + b\ell(x)}{1 + \ell(x)} = F^{-1}\{y[F(b) - F(a)] + F(a)\} \\
\Leftrightarrow & \ell(x) = \frac{a - F^{-1}\{y[F(b) - F(a)] + F(a)\}}{F^{-1}\{y[F(b) - F(a)] + F(a)\} - b} \\
\Leftrightarrow & x = \ell^{-1}\left[\frac{a - F^{-1}\{y[F(b) - F(a)] + F(a)\}}{F^{-1}\{y[F(b) - F(a)] + F(a)\} - b}\right].
\end{aligned}$$

The desired result is obtained. □

Proof of Proposition 4.1 The proof is almost identical to that of Proposition 2.1. It is enough to consider $\lim_{x\to 0} \ell(x) = 0$ instead of $\lim_{x\to -\infty} \ell(x) = 0$. We thus omit the details. □