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# From symmetry to asymmetry with digital tools in mathematics teacher education

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## Highlights:

- Mathematics teacher education.
- Mathematics knowledge for teaching.
- Algebraic equations and inequalities with parameters.
- Asymmetry of generalized Golden Ratios.
- Fibonacci-like polynomials.

**Abstract:** The paper reflects on the author's experience of teaching mathematics education courses to teacher candidates using technology. It demonstrates both explicit and hidden presence of asymmetry among the topics included in the courses and shows what it brings to the study of mathematics for teaching by future teachers of the subject matter. Examples associated with asymmetry deal with pizza sharing, an irrational inequality, a quadratic equation with parameters the loci of which exhibit asymmetry, and asymmetrical modification of Pascal's triangle leading to Fibonacci-like polynomials the roots of which alternate symmetrical and asymmetrical location within a certain interval on the number line. The examples presented in the paper are supported by digital tools including dynamic geometry programs, Wolfram Alpha, Maple, a spreadsheet and the graphics of Microsoft Word. The paper argues for the importance of the modern-day computational tools in revealing conceptually rich structure of mathematics, including the interplay between symmetry and asymmetry, to future K-12 teachers of the subject matter. Several reflective comments of the teachers on their learning experiences to teach mathematics conceptually in the technological paradigm are shared and analyzed.

**Keywords:** mathematics education; teacher education; technology; asymmetry; symmetry; algebra; discrete mathematics

## 1. Introduction

The paper is written to reflect on the author's teaching mathematics education courses to K-12 teacher candidates of the United States and Canada. The university where the author works is in the proximity to the border of the two countries thereby enabling the international character of its teacher education programs, especially online programs. This reflection is in response to the call for papers for *Asymmetry* the scope of which includes social sciences to which, according to Dewey [1] and, assuming that "education is a social process" [2], mathematics teacher education belongs. Being motivated by the appearance of a new journal, the author realized that what was taught to prospective teachers of mathematics over the years can be described through the lens of asymmetry and be used by mathematics educators under this new umbrella. In other words, whereas the term asymmetry was not necessarily used by the author in the past, the conceptual bond between symmetrical and not symmetrical (*i.e.*, asymmetrical) images was emphasized any time when a change either in a problem-solving method or in a symbolism of mathematics yielded conceptually significant change in the associated imagery.

With this in mind, the author intends to demonstrate both explicit and hidden presence of asymmetry in the curricula of the courses taught and to highlight this special attribute of a mathematical entity in terms of what it brings to the study of mathematics for teaching [3–5]. This study advances mathematics teacher education as a domain of disciplined inquiry for it "requires looking at mathematics from the point of view of education and also looking at mathematics education with a broad understanding of elementary mathematics" [6]. Such social duality of the didactics of a mathematics classroom implies the need for a teacher acting as a mathematician by seeing the general in the particular [7] and for a mathematician acting as a teacher by "making the results of the work of geniuses available to everyone" [8]. Furthermore, the epistemic duality of the abstractness of the subject matter and the concreteness of its pedagogical essentials can be used by mathematics teacher educators as a research-like experience for their students.

The paper includes several examples associated with asymmetry from different grade levels of mathematics curricula. Some elementary school examples will show how moving from the context of symmetry to that of asymmetry enables a more effective solution of a problem. Other examples will demonstrate the place for asymmetry in the secondary school algebra enhanced by the modern-day digital tools. Such tools allow for interactive demonstration of the relationship between two-variable equations or inequalities (when at least one variable is a parameter) and their geometric images (loci), symmetrical in one case and asymmetrical in another case, to allow for symmetry to bifurcate into asymmetry and vice versa. Yet another example will show how turning the symmetry of classic Pascal's triangle into the asymmetry of its entries provides mathematics for teaching with "springboards to the advanced mathematics … topics ranging from Fibonacci numbers to continued fractions" [3] and results in the discovery of new knowledge not only from a mathematics education perspective but being new for mathematics itself. This knowledge includes new type of polynomials associated with Fibonacci numbers the real roots of which, depending on the rank of the famous numbers, alternate their symmetrical and asymmetrical location within a certain interval and serve as springboards into the phenomenon of a generalized Golden Ratio with its own attribute that can be seen as asymmetry. Experiencing serendipitous transition from asymmetric modification of the entries of the well-known Pascal's triangle to the little known one-variable polynomials allows future teachers to appreciate that "knowledge of established mathematics is inseparable from knowledge of how mathematics is established … interconnections among ideas, and the analogies and images that have come to be associated with different principles" [4]. Knowing how such images and principles can be revealed to their own students using technology is critical for the modern-day successful mathematics teaching.

In Common Core State Standards [9], which has been the major educational document in the United States since its appearance in 2010, the word symmetry is mentioned already in grade one. It can also be found in the Standards in the contexts of functions and trigonometry in the high school curriculum. Whereas the word asymmetry is not used in the document, using the word different in the teaching of mathematics allows one to understand not only how symmetry can follow from asymmetry but also how asymmetry gives birth to symmetry through certain modifications of the former. There are problems, already discussed at the elementary level when students discover that an action on objects as simple as paper folding leads to seemingly asymmetric constructions. Yet another simple action such as applying scissors to paper allows one seeing that the reflective symmetry is not the only symmetry mathematics deals with. Indeed, the task of folding a bank paper about a diagonal, shows asymmetry of two triangles about the diagonal, something that, in fact, is the case of rotational symmetry. Therefore, already at the early elementary level, mathematical knowledge for teaching should allow one not only to recognize the importance of "the tasks involved in teaching and the mathematical demands of these tasks" [5] but, by seeing the ordinary through the meticulous lens, to be able to provide students with collateral learning opportunities [10].

## 2. Materials and methods

Two types of materials have been used by the author when working on this paper. The first type is digital, the so-called "mathematical action technologies" [11]. These technologies include computational knowledge engine Wolfram Alpha developed by Wolfram Research (www.woframalpha.com, accessed on 1 August 2024); the mathematical software Maple [12] used by the author for symbolic computations; an electronic spreadsheet used by the author as a tool for numeric modeling; computer algebra system the Graphing Calculator produced by Pacific Tech [13], that supported the construction of loci defined by algebraic inequalities with parameters; GeoGebra [14], the Geometer's Sketchpad [15], and MS Word used for construction of images at the lower elementary level. The second type of materials used by the author included teaching and learning mathematics standards and recommendations for teaching used in the United States [10,16] and Canada [17]. The standards call for fostering mathematical reasoning in the technological paradigm and using computer-generated representations of concepts when solving mathematical problems.

Methods specific for mathematics teacher education used by the author include computer-based mathematics education, standards-based mathematics, and problem solving. In particular, those methods are conducive to presenting teacher candidates with "connections between seemingly unrelated concepts" [16]. Teacher candidates learn how to think computationally by "expressing problems in such a way that their solutions can be reached using computational steps and algorithms" [17] and learning "to express the computation in general terms, abstracting from specific instances" [10]. Many secondary mathematics teacher preparation programs in the United States offer courses "that include topics such as … finite difference equations, iteration and recursion … and computer programming" [16]. These topics underpin several computational algorithms used in this paper.

## 3. Asymmetry at the lower elementary level

In the United States, Common Core State Standards [9] expect students in grades K-2 to "describe basic two-dimensional shapes … presented in a variety of ways … to construct more complex shapes …

recognize them from different perspectives and orientations … determine how they are alike and different … equal shares of identical wholes need not have the same shape”. In Canada [17], second graders are expected to “compose and decompose two-dimensional shapes … to create larger shapes (composing) or broken into smaller shapes (decomposing) … some triangles have three lines of symmetry, some have one line of symmetry, and some have no lines of symmetry”. As a way of addressing these expectations, teacher candidates (the author’s students) are advised that young children can be given symmetrical shapes (e.g., isosceles triangles and trapezoids, rectangles, rhombuses, *etc.*) and asked to put them together to see if the combination of two (or more) symmetrical figures (with at least one side length being equal among the shapes) retains the attribute of symmetry or not (*i.e.*, becomes asymmetrical). Children can be asked to trace a symmetrical figure on a piece of paper and then use paper folding, holes punching, or scissors to demonstrate the relationship between the two parts. By doing so, one can learn that in the case of a single symmetrical figure, its partitioning in two figures through paper folding results in asymmetrical parts more often than in symmetrical ones. Whereas any regular polygon with $n$ sides has $n$ lines of reflective symmetry, any other line passing through the polygon is a line of asymmetry for the polygon. One can also discover that a combination of such lines of asymmetry can create a symmetrical shape (or shapes). The use of the modern-day digital tools such as the Graphing Calculator, GeoGebra, Geometer’s Sketchpad, and even Microsoft Word makes it possible to visualize the interplay between symmetry and asymmetry when solving a mathematical problem. In algebra, such interplay may be controlled by parameters. In geometry, as can be seen in the Egyptian papyrus rolls [18], whereas symmetry can be recognized visually already in the architecture of the Neolithic age [19], it was not used by the scribes in finding areas of isosceles triangles with the base being much smaller than a lateral side. In fact, the latter being close in size to the altitude which, as the line of symmetry, bisects the former, is used as one of the factors when calculating area. On the other hand, according to Luchins & Luchins [20], ancient Egyptians knew how to find areas of asymmetrical shapes by extending them to form rectangles (e.g., Figure 1). This aspect of geometry does contribute to mathematics knowledge for teaching with “insights into the historical emergence of core concepts” [4].

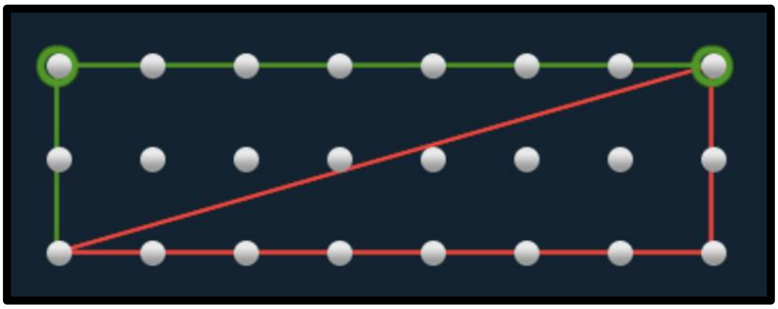


**Figure 1.** Extending asymmetrical right triangle to rectangle in finding area of the latter.

## 4. Asymmetry at the upper elementary level

Fractions have been always considered difficult to teach and learn (e.g., [21–23]). Using the context of pizza sharing makes it possible to create an environment within which one deals with fractions visually without using any mathematics other than counting and measuring. In such an environment, dividing fairly $m$ identical circular pizzas among $n$ people, $n > m$, with the minimal number of slices using a pizza wheel can result in either mutually asymmetrical or symmetrical slices. In other words, visualization allows one to see the results of division of pizzas through the lens of symmetry and asymmetry. Consider the case $m = 3$ and $n = 5$, that is, dividing three pizzas among five people. Symmetrical division, *i.e.*, dividing each pizza in five identical slices, is what typically first comes to mind (Figure 2). However, the number of pieces, 15, is not the minimal. To reduce the number of slices, one can cut each pizza in half and then divide the remaining half in five equal pieces (Figure 3). In that way, the third pizza would be cut in asymmetrical slices. Although this action does reduce the number of slices by five, it is still possible to have a smaller number of slices. In fact, the first symmetrical partition of pizzas into 15 slices (Figure 2) can prompt the division with the minimal number of slices, 7, when two pizzas are divided in two unequal slices each and one pizza is divided in three pieces, two of which are identical (Figure 4). One can say that whereas the slices of the third pizza are globally asymmetric, some of them are locally symmetric, both reflectively and rotationally. Note that the fairness of the division of pizzas to get 7 slices can be verified through measurement rather than numerically by using fractions. That is, only counting and measuring was involved in this activity through which both symmetry and asymmetry can be discussed. In fact, measurement is a more effective way of verifying symmetry of parts of the whole than through a recourse to fractions as even a rectangle may be cut into two halves to demonstrate asymmetry (see, e.g., Figure 1).

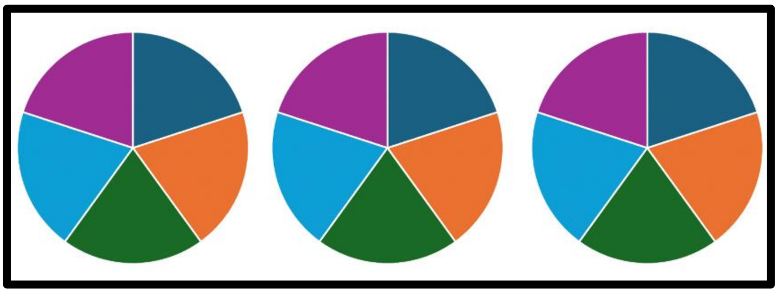


**Figure 2.** Dividing 3 pizzas among 5 people with 15 slices (MS Word).

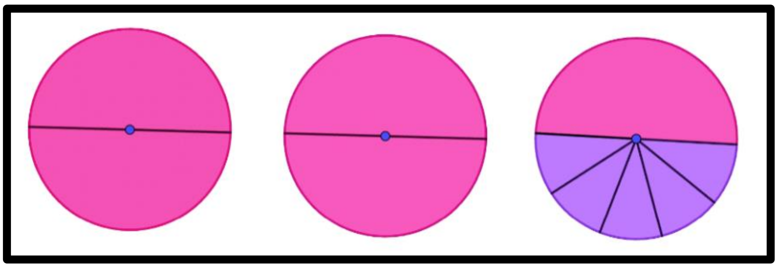


**Figure 3.** Dividing 3 pizzas among 5 people with 10 slices (GeoGebra).

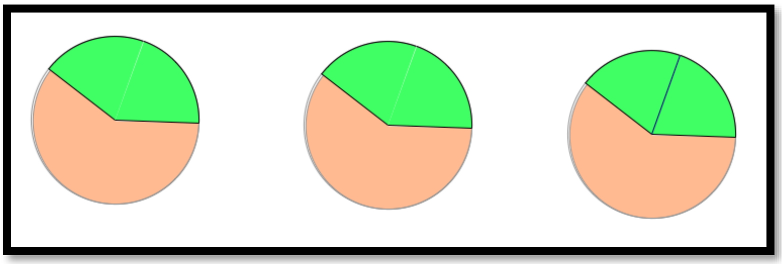


**Figure 4.** Dividing 3 pizzas among 5 people with 7 slices (Geometer's Sketchpad).

## 5. Asymmetry in advanced high school algebra

High school algebra provides many opportunities to talk about symmetry and asymmetry on the spectrum from the location of roots of quadratic equations about a certain point to the images of the loci of inequalities with a parameter. Digital fabrication [24,25] made possible by the Graphing Calculator allows for constructing images of loci of different equations and inequalities in two variables. Connecting algebra to geometry allows one to learn how visual is controlled by symbolic and vice versa, how certain changes in geometric shapes create conditions for analytic description of their boundaries. To illustrate, consider the inequality

$$\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}} \le \sqrt{2} \quad (1)$$

which has to be solved in $x$ with respect to parameter $a$. In the digital era, solution can include graphic construction of the locus of inequality (1) and analytic description of the locus by entering the inequality into the input boxes of graphing and symbolic software tools. If the locus constructed is asymmetrical, one can use different modifications of inequality (1) to make the extension of the original locus symmetrical. Furthermore, one can demonstrate the sensitivity of the symmetry of images to their analytical description.

The first step is to construct the locus of inequality (1) as shown in Figure 5. The image of the locus requires analytical clarification as the graphing software, while being capable of carrying out the exact construction of the borders of the locus, does not provide their analytic description. However, one can use Wolfram Alpha (Figure 6) in solving inequality (1) to find out that in the plane ($x$, $a$) the curvilinear border of the locus is described by the equation $x = a^2$ and the other two borders, vertical and slanted, are described, respectively, by the equations $x = 0$ and $x = 2a - 1$. Visually, one can see that the locus is asymmetrical. This can also be confirmed by the analytic asymmetry of the equations of the straight and the curvilinear lines. Knowing the equations of the borders, one can use computational triangulation (CT) introduced in [26] between the construction of the locus by the Graphing Calculator and its analytical description by Wolfram Alpha; that is, by using the former tool to trace the borders with no gap or overlap with the locus as shown in Figure 7. As a theoretical construct prompted by the social sciences research [27,28], CT in mathematical problem solving may be seen as a replacement of "the traditional social process of proof" [29] to avoid both subtle and unsubtle errors in the age of technology.

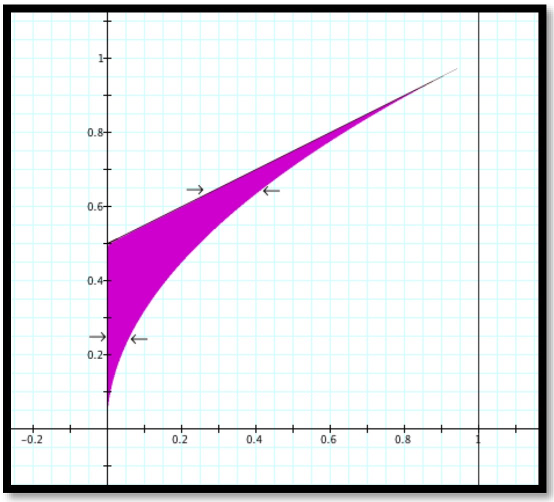



**Figure 5.** The locus of the inequality $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}} \le \sqrt{2}$.

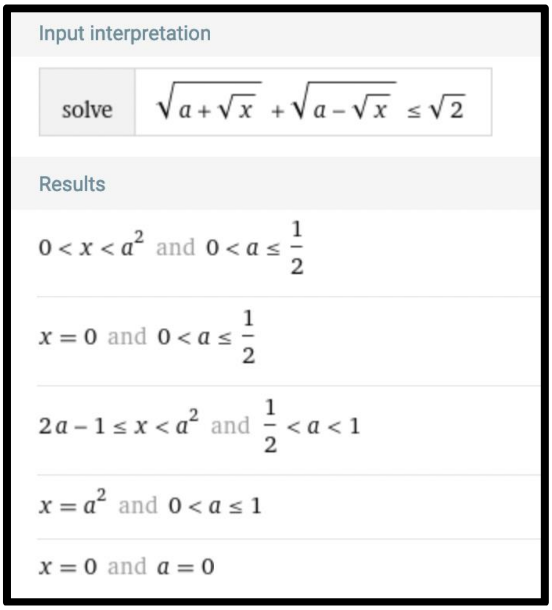



**Figure 6.** Solving inequality (1) by Wolfram Alpha.

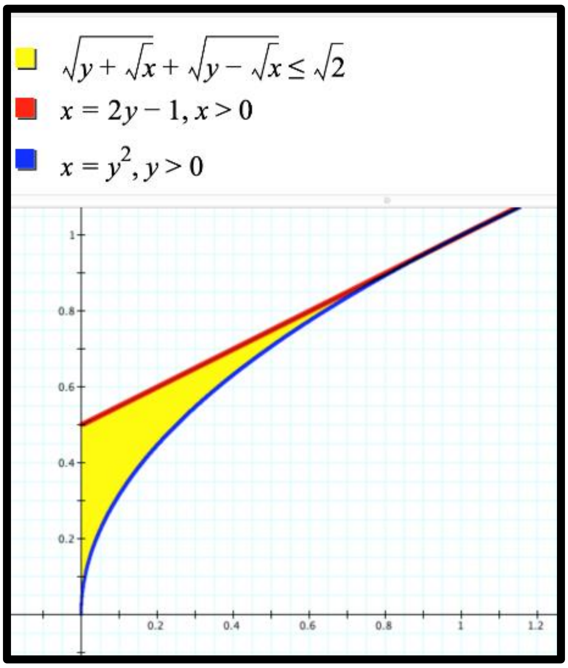



**Figure 7.** CT between Wolfram Alpha's solution of (1) and the Graphing Calculator (*y* = *a*).

Whereas some obvious modifications of inequality (1), such as shown in Figure 8, result in the symmetry of the corresponding locus, Figure 9 (in which *y* = *a* due to the notational constraint of the software) shows a more complicated modification in constructing the reflection of the point (*x*, *a*) in the line $x=2a-1$ yielding the point $((3x-4a-2)/5,(4x-3a+4)/5)$. Therefore, the following modification of inequality (1)

$$\sqrt{\frac{4x-3a+4}{5}+\sqrt{\frac{3x+4a-2}{5}}}+\sqrt{\frac{4x-3a+4}{5}-\sqrt{\frac{3x+4a-2}{5}}}\le\sqrt{2} \quad (2)$$

should have a symmetrical locus (Figure 9).

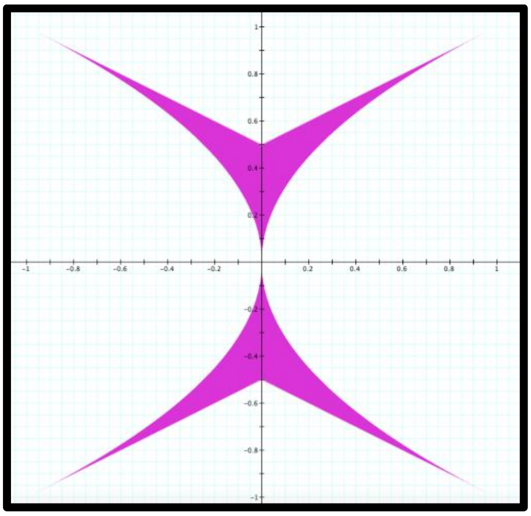


**Figure 8.** The locus of the inequality $\sqrt{|a|+\sqrt{|x|}}+\sqrt{|a|-\sqrt{|x|}} \le \sqrt{2}$ .

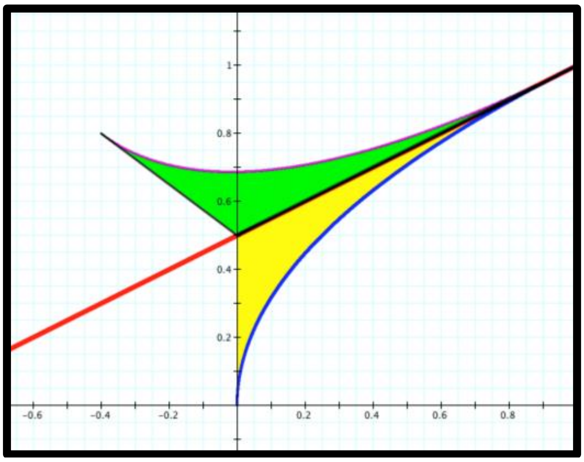


**Figure 9.** From asymmetry of the locus of (1) to reflective symmetry in its slant border.

Figure 10 shows an asymmetrical image resulted from replacing the expression $(4x-3a+4)/5$ by $(4x-3a+4.5)/5$ twice as a modification of inequality (2). This shows the sensitivity of a symmetrical image on its analytical description. While the sensitivity is not very significant from the point of view of an observer, an image in order to be recognized as symmetrical requires its analytic description to be precise. As mentioned by a future secondary mathematics teacher (the author's student), "When variables are viewed in isolation from the totality, they can seem abstract or irrelevant. It really made sense to me with the graph, equation, and coordinates and how those all work together to make the problems visually imaginable." This remark points at the student's appreciation of the concreteness of symmetry *vs.* asymmetry provided by computer graphing in comprehending the abstractness of symbolic reflection of the inequality's locus in a straight line different from a coordinate axis. The focus on "totality" of a particular concept in the above comment indicates that knowledge for teaching mathematics should include teachers' mastery of multiple representations of the concept. At the core of this mastery is facilitating the convergence of the representations to the concreteness of images controlled by the abstractness of variables involved in equations and inequalities.

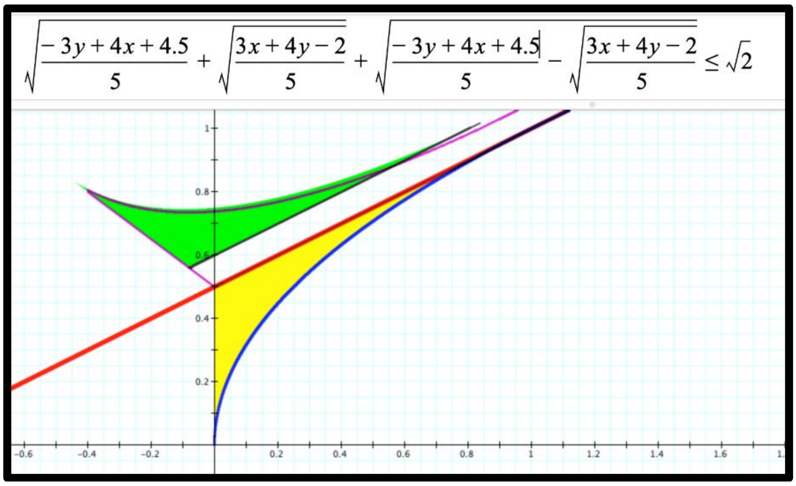



**Figure 10.** Sensitivity of symmetry on analytics: replacing 4 by 4.5 in (2) yields asymmetry.

## 6. Bifurcating from symmetry to asymmetry in the plane of parameters

Consider an artificial word with two letters E and two letters R. There are six permutations of the four letters: EERR, RREE, ERER, RERE, ERRE, and REER. The first two permutations, EERR and RREE, display asymmetry in the string of four letters. The other four permutations display symmetry. One of them is ERRE. Let us assign to the letters R and E the following meaning: R is a real root of a quadratic function and E is an end point of an interval on the number line. The permutation EERR means that the roots of the function are located to the right of the interval; the permutation RREE means that the roots of the function are located to the left of the interval. Let the quadratic function has the form $f(x)=x^2+bx+c$, where $b^2-4c>0$, and the interval is $(-n,n)$. The cases of EERR and RREE are shown in Figures 11 and 12, respectively. The case ERRE is shown in Figure 13. Whereas two roots of the quadratic function $f(x)$ are always symmetrical about the point $x=-b/2$ and the endpoints of the interval $(-n,n)$ are always symmetrical about the origin, the quadruple of the points, in general, represents an asymmetric combination. In the specific case when the distance between the roots coincides with the length of the interval, we have asymmetry turned into symmetry. That is, if $x_1=\left(-b+\sqrt{b^2-4c}\right)/2$ and $x_2=\left(-b-\sqrt{b^2-4c}\right)/2$, $x_1>x_2$, we have $x_1-x_2=\sqrt{b^2-4c}$. So, in the case $\sqrt{b^2-4c}=2n$ we have a symmetrical location of the quadruple of points on the number line. More specifically, the pairs of points $(-b-2n)/2$ and $n$ as well as $(-b+2n)/2$ and $-n$ are symmetrical about the point $x=-b/4$.

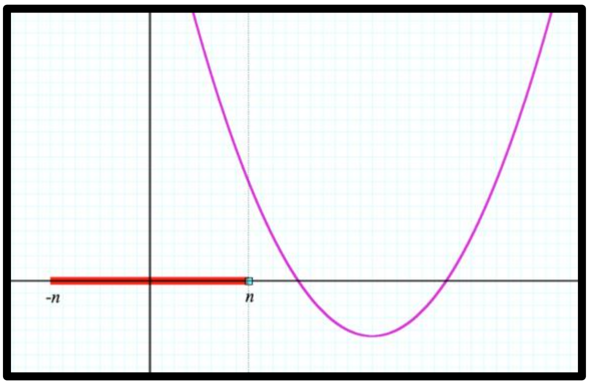



**Figure 11.** The asymmetry of EERR in the coordinate plane.

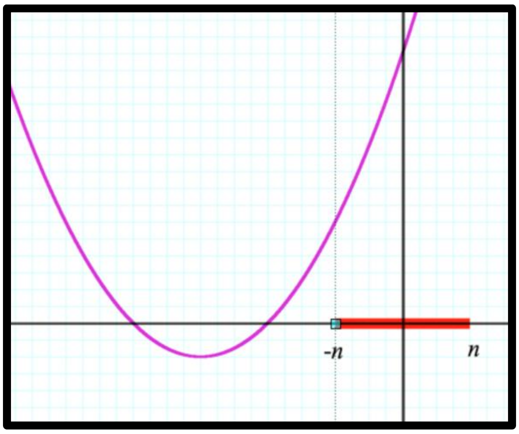



**Figure 12.** The asymmetry of RREE in the coordinate plane.

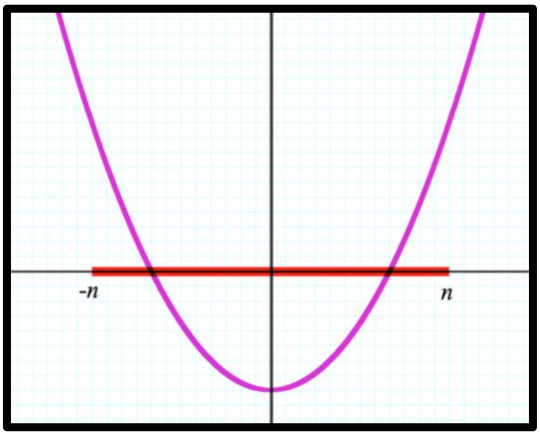



**Figure 13.** The symmetry of ERRE in the coordinate plane.

The next step in the exploration is to see whether the asymmetry of letters in the strings EERR and RREE as well the asymmetry of the quadruples of points shown in Figures 11 and 12 retain this attribute when the cases are expressed in the form of loci in the plane of coefficients ($c$, $b$) of the quadratic function for a specific value of $n$. To begin, consider the diagram of Figure 11. In addition to the inequalities $f(n) > 0$ and $b^2 - 4c > 0$, the vertex of the parabola $y = x^2 + bx + c$ must be located to the right of the point $x = n$. Hence

$$n^2 + bn + c > 0, b^2 - 4c > 0, -\frac{b}{2} > n \qquad (3)$$

The locus of the system of inequalities (3) in the case $n = 1$ is shown in Figure 14. The lines $b^2 - 4c = 0$ and $1 + b + c = 0$ meet at the point (1, −2). That is, the region EERR in the coordinate plane ($c$, $b$) is located to the right of the line $c = 1$, below the line $b^2 - 4c = 0$, and above the line $1 + b + c = 0$. One can see that the region EERR (Figure 14) retains the attribute of asymmetry observed for the string of letters and for the quadruple of points in Figure 11.

Consider now the diagram of Figure 12. In addition to the inequalities $f(-n) > 0$ and $b^2 - 4c > 0$, the vertex of the parabola $y = x^2 + bx + c$ must be located to the left of the point $x = -n$. Hence

$$n^2 - bn + c > 0, b^2 - 4c > 0, -\frac{b}{2} < -n \qquad (4)$$

The locus of inequalities (4) in the case $n = 1$ is shown in Figure 15. The lines $b^2 - 4c = 0$ and $1 - b + c = 0$ meet at the point (1, 2). That is, the region RREE in the coordinate plane ($c$, $b$) is located to the right of the line $c = 1$, above the line $b^2 - 4c = 0$, and below the line $1 - b + c = 0$. One can see that the region RREE (Figure 15) retains the attribute of asymmetry observed for the string of the four letters and for the quadruple of points in Figure 12.

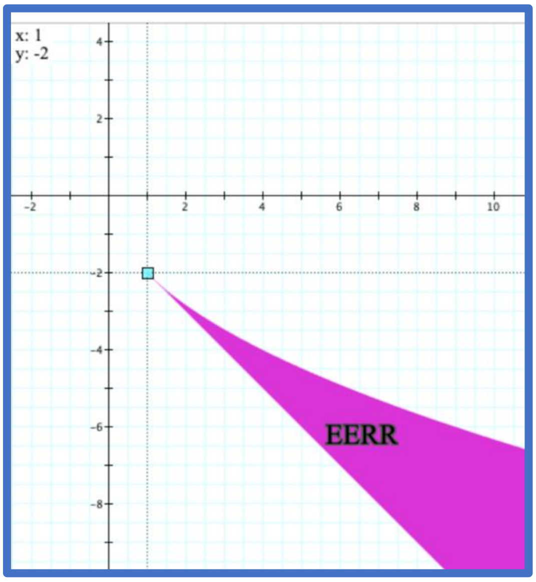



**Figure 14.** The asymmetry of EERR in the plane of parameters.

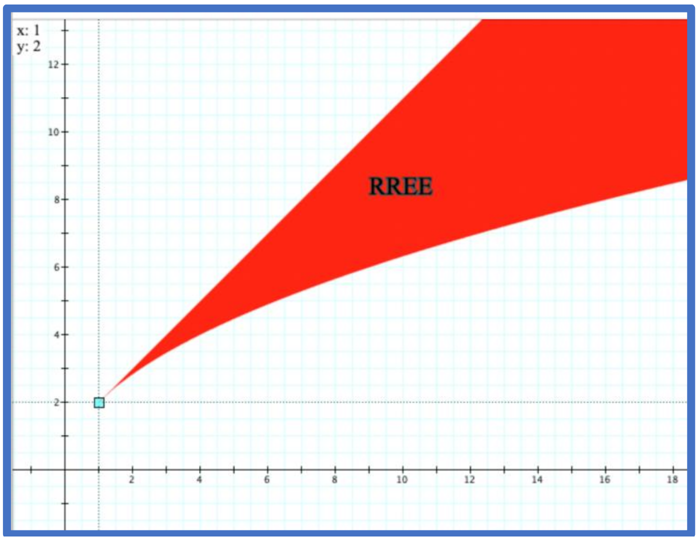



**Figure 15.** The asymmetry of the RREE in the plane of parameters as defined by (4) with $n = 1$.

Now, consider the diagram of Figure 13, the case of ERRE, with the symmetry of the four letters in the string and the symmetry of the corresponding quadruple of points. In addition to the inequality $b^2 - 4c > 0$ the diagram suggests the inequalities $f(\pm n) > 0$ and, to allow the vertex of the parabola $y = x^2 + bx + c$ to belong to the interval $(-n, n)$, requires the inequality $\left|b/2\right| < n$. When $n = 1$, the system of inequalities

$$1 + b + c > 0, 1 - b + c > 0, b^2 - 4c > 0, |b| < 2 \quad (5)$$

defines the region in the plane ($c$, $b$) of the parameters each point of which provides the quadratic function $f(x) = x^2 + bx + c$ with both roots located within the corresponding interval. The locus of the system of inequalities (5) is shown in Figure 16 to demonstrate the following: in the plane of parameters ($c$, $b$) the region, each point of which provides the quadratic function $f(x) = x^2 + bx + c$ with both roots located inside the interval ($-n$, $n$), retains the attribute of symmetry observed for the string of letters ERRE and for the quadruple of points on the number line (Figure 13). The region is located to the left of the line $c = 1$ and between the lines $1+b+c=0, 1-b+c=0, b^2-4c=0$. One can also observe in the diagram of Figure 16 how symmetry bifurcates into asymmetry (and vice versa) as the parameters of the quadratic function $f(x) = x^2 + bx + c$ pass through the points (1, 2) and (1, −2). Introducing $n$ as the third parameter turns the last two points into ($n$, $2n$) and ($n$, $-2n$). One can see that the regions RREE and EERR, while being asymmetric individually in the plane ($c$, $b$), are mutually symmetric about the axis $b = 0$ in the ($c$, $b$)-plane. In particular, when $c = n^2$ and $b = 2n$ we have $n = \sqrt{b^2 - 4c} = 0$; that is, the interval ($-n$, $n$) converges into the origin and the vertex of the parabola resides on the vertical axis in the coordinate plane. That is, bifurcation of symmetry into asymmetry (and vice versa) takes place when the parabola $y = x^2 + bx + c$ is symmetrical about the $y$-axis. Finally, as shown in Figure 17 towards verifying the correctness of the constructions across parametric and coordinate planes, selecting the point (5, −5) in the ($c$, $b$)-plane (Figure 17, left) and graphing the parabola $y = x^2 - 5x + 5$, provides the EERR case in the ($x$, $y$)-plane with an asymmetric quadruple of points (Figure 17, right).

To conclude this section, the following two reflections on a capstone project "Exploring quadratic equations depending on parameters" completed by teacher candidates in the secondary mathematics education course taught by the author can be shared. As one candidate put it, "For most of the 9–12 mathematics curriculum, quadratic equations, their roots and graphs are an integral part. The biggest oversight I see is that teachers feel these topics need to be taught in isolation. When in fact they can all be used together to help students make connections. A student who understands the purpose of the procedure, will never forget it. When teaching the material conceptually, it pays off in the end." Another candidate believes that "effective mathematics teacher education programs should take more advantage of technological apparatus, which will allow more grade-appropriate advanced mathematical ideas. This will allow movement from the more formal teaching to a constructivist approach which is more attuned to formal reasoning."

These comments indicate the candidates' dynamic personalities who evince sincere efforts to gain knowledge for teaching by exploring the existing varieties of types of location of real roots of quadratic equations with parameters as the core conceptual idea of the capstone project. Including such explorations in a technology-enhanced mathematics teacher education course allows for the growth of "dynamic individual understandings … the conceptual sophistication needed … to structure rich learning experiences" [4] of the learners of mathematics, whatever their immediate learning goals and aspirations are.

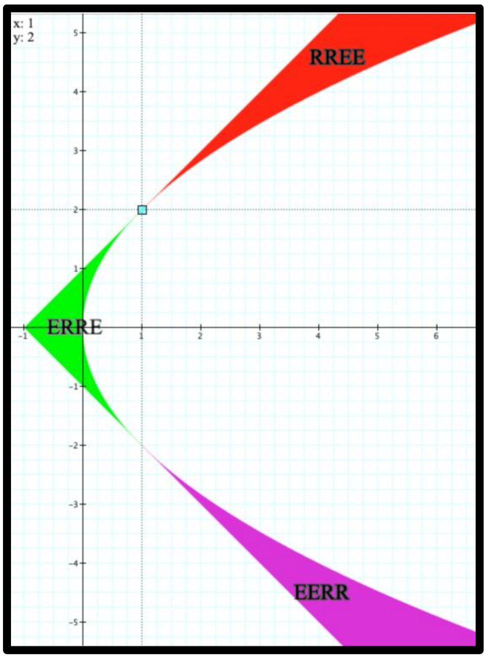



**Figure 16.** ERRE bifurcates into RREE and EERR.

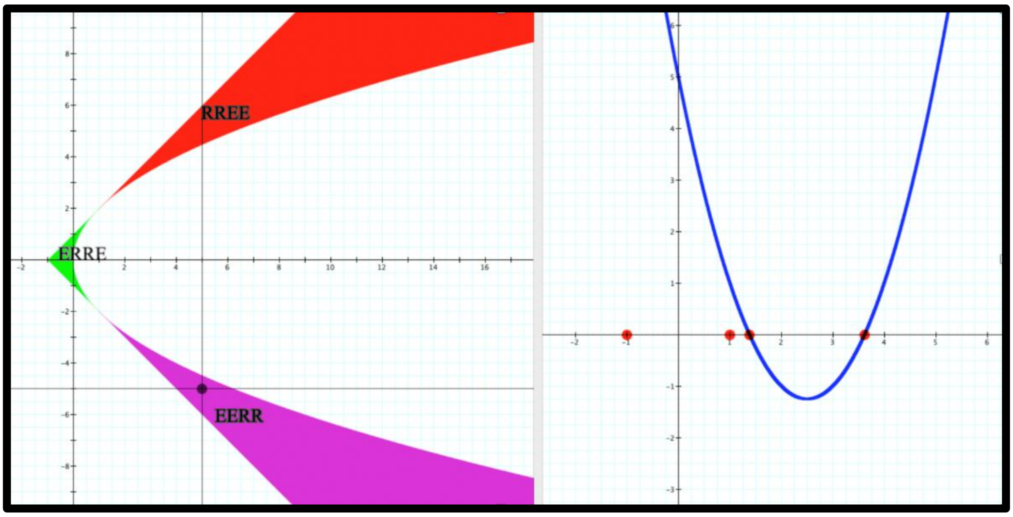



**Figure 17.** The point (5, 5) on the $(c, b)$-plane provides asymmetry on the $(x, y)$-plane.

## 7. Asymmetry of entries of Pascal's triangle yields Fibonacci-like polynomials

Consider Pascal's triangle shown in Figure 18. One can see that the entries of Pascal's triangle, being binomial coefficients, are symmetrical within each row. The top right-bottom left diagonals of the triangle consist, respectively, of ones, natural numbers, triangular numbers, triangular pyramidal numbers, pentatope numbers, and so on. Figure 19 shows asymmetrical rearrangement of those numbers when the diagonal with ones forms the first column, the diagonal with natural numbers forms the second column shifted down about the first one by two rows, the diagonal with triangular numbers becomes the third column shifted down about the second one by two rows, the diagonal with triangular pyramidal numbers forms the fourth column shifted down about the third one by two rows, and so on.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1 | | | | | | |
| | | | | | 1 | | 1 | | | | | |
| | | | | 1 | | 2 | | 1 | | | | |
| | | | 1 | | 3 | | 3 | | 1 | | | |
| | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

**Figure 18.** Classic Pascal's triangle is symmetrical.

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 2 | | | | |
| 1 | 3 | 1 | | | |
| 1 | 4 | 3 | | | |
| 1 | 5 | 6 | 1 | | |
| 1 | 6 | 10 | 4 | | |
| 1 | 7 | 15 | 10 | 1 | |
| 1 | 8 | 21 | 20 | 5 | |
| 1 | 9 | 28 | 35 | 15 | 1 |
| 1 | 10 | 36 | 56 | 35 | 6 |

**Figure 19.** The modified Pascal's triangle is asymmetrical.

An interesting property of numbers appearing in each row of the rearranged Pascal's triangle (Figure 19), in addition to having their sums in each row equal to consecutive Fibonacci numbers, is that using those numbers as coefficients of one-variable polynomials, results in polynomials with real roots only, all located within the interval $(-4, 0)$. Furthermore, out of two polynomials of the same degree, one polynomial has symmetrical location of roots, and another polynomial has asymmetrical location of roots. These observations would not be possible without the use of digital technology. For example, the numbers 1, 6, 10, 4, the sum of which is the 8th Fibonacci number 21, form the polynomial $f(x) = x^3 + 6x^2 + 10x + 4$ the graph of which (Figure 20, left) intersects the interval $(-4, 0)$ three times. The two roots, $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$ are symmetrical about the third root $x = -2$. At the same time, the numbers 1, 5, 6, 1, the sum of which is the 7th Fibonacci number 13, form the polynomial $f(x) = x^3 + 5x^2 + 6x + 1$ the graph of which (Figure 20, right) intersects the interval $(-4, 0)$ three times as well, yet the points of intersection are not symmetrical as the following relations demonstrate:

$$-1.555 - (-3.247) = 1.642 \neq 1.357 = -0.198 - (-1.555).$$

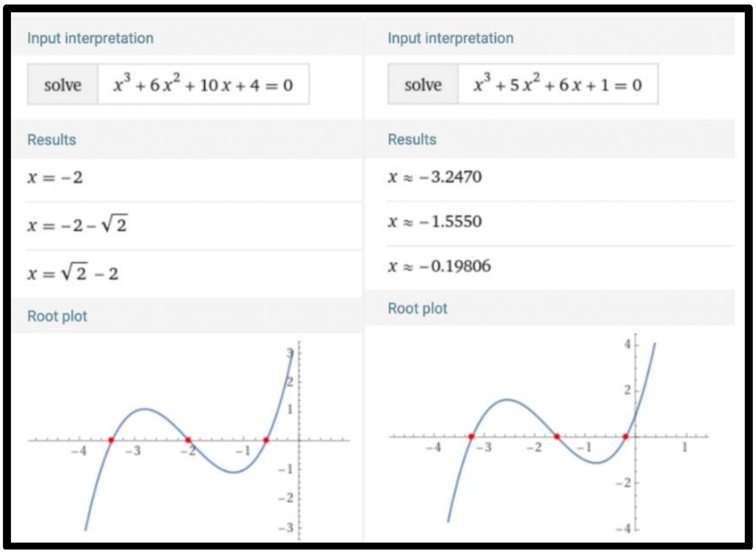



**Figure 20.** Fibonacci-like polynomials of degree three and their roots.

Likewise, the numbers 1, 8, 21, 20, 5, the sum of which is the 10th Fibonacci number 55, form the polynomial $f(x)=x^4+8x^3+21x^2+20x+5$ the graph of which (Figure 21, left) intersects the interval $(-4,0)$ four times. The roots, $x=\left(-5-\sqrt{5}\right)/2$ and $x=\left(\sqrt{5}-3\right)/2$ are symmetrical about the point $x=-2$, as $-2-\left(-5-\sqrt{5}\right)/2=\left(1+\sqrt{5}\right)/2$ and $\left(\sqrt{5}-3\right)/2-(-2)=\left(1+\sqrt{5}\right)/2$. The same is true for another pair of roots. At the same time, the numbers 1, 7, 15, 10, 1, the sum of which is the 9th Fibonacci number 34, form the polynomial $f(x)=x^4+7x^3+15x^2+10x+1$ the graph of which (Figure 21, right) intersects the interval (−4, 0) four times as well, yet the points of intersection clearly display asymmetrical location.

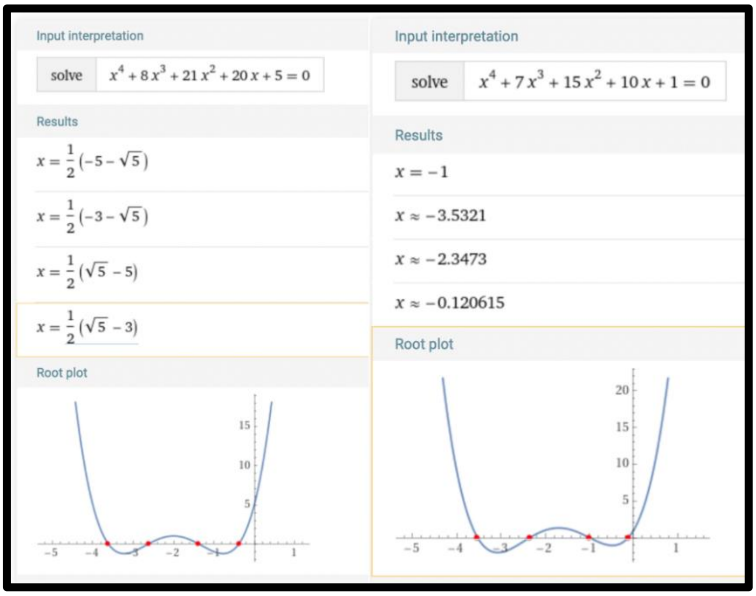



**Figure 21.** Fibonacci-like polynomials of degree four and their roots.

Figure 22 displays two polynomials of degree five, with coefficients borrowed from the last two rows of the modified Pascal's triangle of Figure 19, one of which, $f(x)=x^5+10x^4+36x^3+56x^2+35x+6$, with the sum of coefficients equal to the 12th Fibonacci number 144, has symmetrical intersection of the interval (−4, 0) in five points; another one, $f(x)=x^5+9x^4+28x^3+35x^2+15x+1$, with the sum of coefficients equal to the 11th Fibonacci number 89, has asymmetrical intersection of the interval (−4, 0) in five points. All such polynomials with coefficients from the rows of the rearranged Pascal's triangle of Figure 19 are called Fibonacci-like polynomials [30]. The use of the word like in the name of the polynomials is due to other types of polynomials associated with Fibonacci numbers; for example, Fibonacci polynomials (e.g., [31]) and Catalan's Fibonacci polynomials [32].

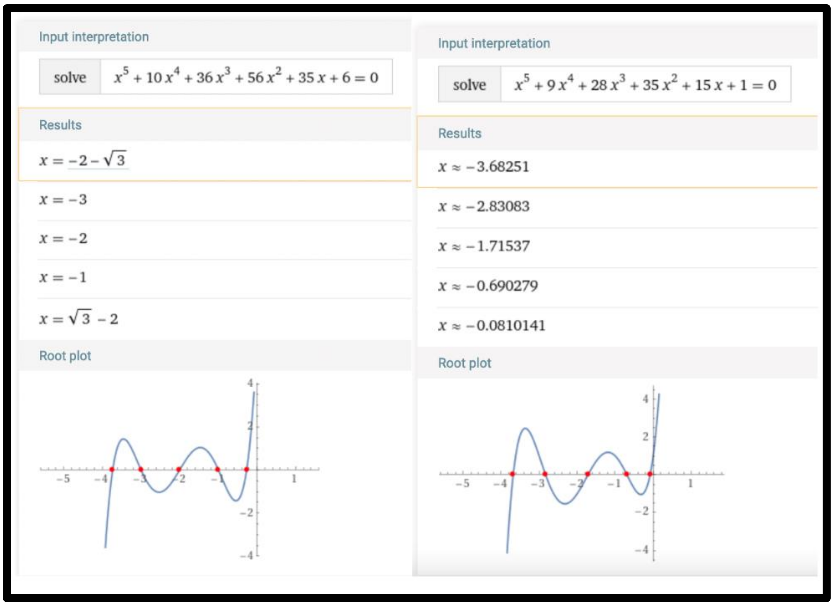



**Figure 22.** Fibonacci-like polynomials of degree five and their roots.

## 8. Difference equations and asymmetry of generalized Golden Ratios

The roots of Fibonacci-like polynomials, in addition to alternating their type of location within the interval (−4, 0), *i.e.*, symmetrical *vs.* asymmetrical for each degree, are responsible for entrusting the coefficients of the second order linear difference equation

$$f_{n+1}=af_n+bf_{n-1},\ f_1=1,\ f_2=2 \tag{6}$$

defining Fibonacci numbers in the case *a* = *b* = 1, with relations that provide cyclic behavior of the ratios $f_{n+1}/f_n$ as *n* increases, and the larger the degree of a Fibonacci-like polynomial, the larger is the length of the corresponding cycle. To clarify, let $r_n=f_n/f_{n-1}$. Then it follows from equation (6) that

$$r_{n+1}=a+\frac{b}{r_n},\ r_1=2 \tag{7}$$

so that

$$r_2=a+\frac{b}{2},r_3=a+\cfrac{b}{a+\cfrac{b}{2}},\ldots,r_8=a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{2}}}}}},r_9=a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{2}}}}}}}.$$

In order for the iterations $r_n$ to form cycles of length eight, the relation $r_9 = r_1$, involving the continued fraction, should hold true. In other words, one has to solve the equation $r_9 = 2$ (see the use of Maple in Figure 23). Setting $x = a^2/b$ yields the third-degree polynomial $x^3 + 6x^2 + 10x + 4$ which is exactly the Fibonacci-like polynomial with the (symmetrically located) roots shown in Figure 20 (left).

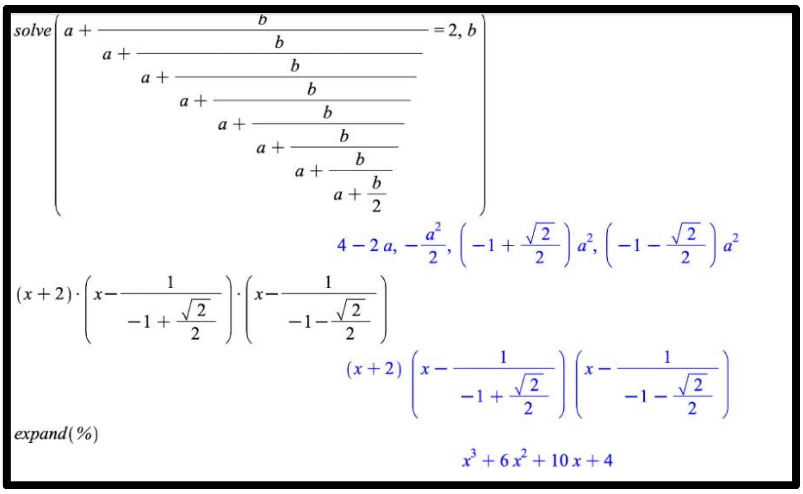
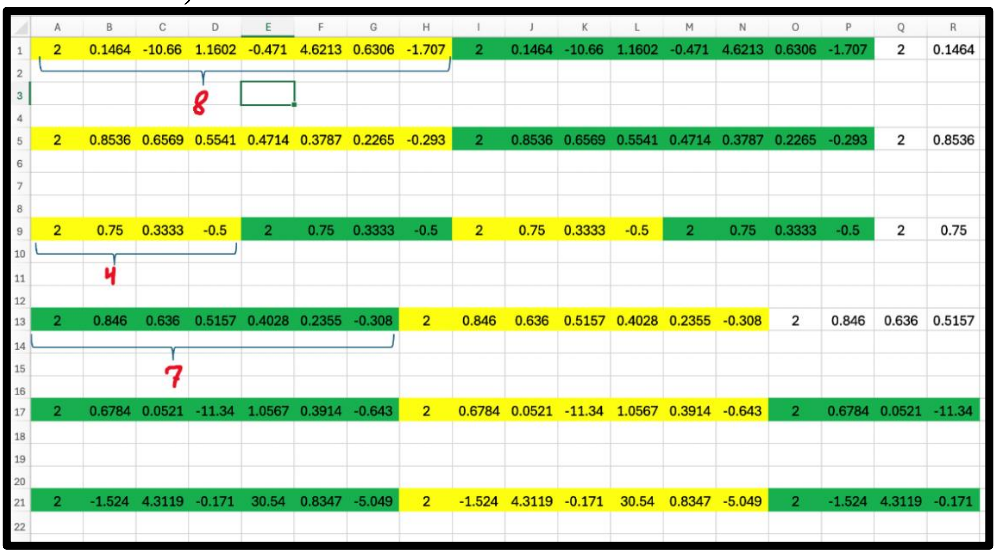



**Figure 23.** From continued fraction defining cycle to Fibonacci-like polynomial.

Selecting its largest root, $x = \sqrt{2} - 2$, one can iterate recursive relation (7) with *a* = 1 within a spreadsheet (Figure 24, row 1) to have a (proper) cycle of length eight. Likewise, selecting the smallest root, $x = -\sqrt{2} - 2$, and iterating (7) with *a* = 1 within a spreadsheet (Figure 24, row 5) to have a (proper) cycle of length eight. However, selecting the root $x = -2$ about which the above two roots are symmetrical, and iterating (7) with *a* = 1 within a spreadsheet (Figure 24, row 9) yields a (proper) cycle of length four which is also a trivial cycle of length eight.

Alternatively, one can take the smallest root, $a^2/b = x \cong -3.2469796037$, out of three asymmetrically located roots of the Fibonacci-like polynomial $x^3 + 5x^2 + 6x + 1$ and iterate the recursive relation (7) with *a* = 1 within a spreadsheet to have a proper cycle of length seven (Figure 24, row 13). Likewise, iterating (7) using the other two roots of the last polynomial also yields proper cycles of length seven (Figure 24, rows 17 and 21).



| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.1464 | -10.66 | 1.1602 | -0.471 | 4.6213 | 0.6306 | -1.707 | 2 | 0.1464 | -10.66 | 1.1602 | -0.471 | 4.6213 | 0.6306 | -1.707 | 2 | 0.1464 |
| 2 | | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | | | | |
| 5 | 2 | 0.8536 | 0.6569 | 0.5541 | 0.4714 | 0.3787 | 0.2265 | -0.293 | 2 | 0.8536 | 0.6569 | 0.5541 | 0.4714 | 0.3787 | 0.2265 | -0.293 | 2 | 0.8536 |
| 6 | | | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | | | | | | | |
| 9 | 2 | 0.75 | 0.3333 | -0.5 | 2 | 0.75 | 0.3333 | -0.5 | 2 | 0.75 | 0.3333 | -0.5 | 2 | 0.75 | 0.3333 | -0.5 | 2 | 0.75 |
| 10 | | | | | | | | | | | | | | | | | | |
| 11 | | | | | | | | | | | | | | | | | | |
| 12 | | | | | | | | | | | | | | | | | | |
| 13 | 2 | 0.846 | 0.636 | 0.5157 | 0.4028 | 0.2355 | -0.308 | 2 | 0.846 | 0.636 | 0.5157 | 0.4028 | 0.2355 | -0.308 | 2 | 0.846 | 0.636 | 0.5157 |
| 14 | | | | | | | | | | | | | | | | | | |
| 15 | | | | | | | | | | | | | | | | | | |
| 16 | | | | | | | | | | | | | | | | | | |
| 17 | 2 | 0.6784 | 0.0521 | -11.34 | 1.0567 | 0.3914 | -0.643 | 2 | 0.6784 | 0.0521 | -11.34 | 1.0567 | 0.3914 | -0.643 | 2 | 0.6784 | 0.0521 | -11.34 |
| 18 | | | | | | | | | | | | | | | | | | |
| 19 | | | | | | | | | | | | | | | | | | |
| 20 | | | | | | | | | | | | | | | | | | |
| 21 | 2 | -1.524 | 4.3119 | -0.171 | 30.54 | 0.8347 | -5.049 | 2 | -1.524 | 4.3119 | -0.171 | 30.54 | 0.8347 | -5.049 | 2 | -1.524 | 4.3119 | -0.171 |
| 22 | | | | | | | | | | | | | | | | | | |



**Figure 24.** Using a spreadsheet to demonstrate proper and trivial cycles.

In general, asymmetrically located within the interval (−4, 0) roots of Fibonacci-like polynomials generate only proper cycles of prime number length and symmetrically located within the interval (−4, 0) roots of Fibonacci-like polynomials generate both trivial and proper cycles of even length. However, cycles of an odd length can also be comprised of cycles the length of which is a divisor of an odd number. For example, the Fibonacci-like polynomial $x^4 + 7x^3 + 15x^2 + 10x + 1$ has four roots, three of which generate proper cycles of length nine and one root, $x = -1$, generates a trivial nine cycle comprised of three cycles of length three. Such cycles, proper and trivial, are called generalized Golden Ratios [30]. To connect cycles to asymmetry, note than a proper cycle of any length represents a string of numbers all of which are different from each other and thereby they may not be symmetrical. That is, all proper cycles, by definition, reveal asymmetry. A cycle the length of which is not a prime number may consist of several cycles that are proper cycles of a smaller length. For example, as was mentioned above, a string of three cycles of length three represents a trivial cycle of length nine which is symmetric. At the same time, all cycles of length seven are asymmetric.

## 9. Results

The paper shared several examples from mathematics teacher education courses taught by the author in which asymmetry as a distinctive attribute of a mathematical entity is present. The examples were not necessarily designed to introduce asymmetry through the courses but its hidden presence in the activities offered to teacher candidates called for looking at education through the lens of mathematics. In doing so, one can learn that not only symmetry, a concept already taught at the lower elementary level, but its antithesis has value, and it is not just something where there is no symmetry, but also a thing that underpins many useful applications of mathematics. Knowing such applications constitutes a valuable part of mathematics knowledge for teaching. It appears that the modern-day students are interested in mathematical studies when they are confident of the usefulness of the material taught to them. In turn, the usefulness implies the applicability in the context of concrete situations.

The paper demonstrated that many topics associated with asymmetry can be found across K-12 mathematics curriculum, especially in the presence of the modern-day physical and digital educational tools. Already a geoboard [33] provides an environment for young students' learning about asymmetry as a "fall for parity" [34] by partitioning a geometric shape as simple as a rectangle in two visually distinct (*i.e.*, asymmetrical) parts using rubber bands (real or virtual). Without using a geoboard, the students may be asked to draw. As one prospective teacher reflected on observing sixth graders in a classroom where "manipulatives are not provided … even for a geometry lesson. Instead, students have to draw. But they could not draw two rectangles the same size and when they did draw them same size, they could not split them in two equal sections." In other words, the sixth graders drew asymmetrical shapes without realizing, most likely along with their teacher, what was going on mathematically. In that way, the use of educational technology, such as the geoboard, supports conceptual understanding that shapes may or may not be symmetrical because, according to Gestalt psychology, "many phenomena of experience are variations organized around Prägnanzstufen phases of clear-cut structure … that an angle of $93°$ is not seen as an entity in its own right but as a "bad" right angle" [35]. This quote might suggest that in the context of Gestalt psychology one could see asymmetry not as an attribute "in its own right" but, simply put, as a "bad" symmetry.

The example of pizza sharing demonstrated the presence of both symmetry and asymmetry in a real-life situation which was resolved more efficiently due to the latter being unconsciously applied. Indeed, the intent of the situation was not to talk about this attribute but only to acknowledge its presence as a reflection on the final solution. One may argue that the effectiveness of mathematics often becomes clear only after its intuitive and technologically enhanced experimental applications, without users' (teacher candidates included) conscious awareness of the role of the concepts involved. Such role may become evident after the application of mathematics "will be tried out, validated, or rejected … following the diversity of the situations and projects the users set for themselves" [36]. Whereas symmetry in real life provides pleasant visualization, agreeable balance, and harmonious proportion [37], asymmetry can be found in nature and described towards the refutation of the conservation of parity [34].

The example of one-variable irrational inequality with a parameter demonstrated how the symbolic controls the visual and how one's understanding of the algebraic nature of this control allows for a digital fabrication of asymmetrical images in the form of the locus of an inequality in the variable *vs.* parameter plane. This example showed different ways of turning asymmetry of an image into an extended image that provides either symmetry or asymmetry. Of course, a modification of analytics to keep or to have an asymmetry is rather simple. So, the example was designed to provide some ideas about transition from asymmetry of a locus to a symmetrical image and then, with a slight modification of the latter to show how the attribute of symmetry depends on the precision of analytics. That is, digital fabrication of a symmetrical design requires good understanding of algebraic symbolism and its connection to geometric images.

Many problems of science and engineering require knowledge of roots of one-variable polynomials and their location in the coordinate plane [38–40]. Investigating quadratic equations with parameters enriches secondary mathematics teacher education courses with explorations redolent of real research experience in STEM (science, technology, engineering, mathematics) fields that teacher candidates need at least to know about. The example about the location of real roots of a quadratic trinomial about an interval on the number line demonstrated that in the plane of parameters to the right of the vertical axis, a symmetrical outcome ERER is very small in comparison with asymmetrical outcomes, RREE and EERR. Whereas this conclusion is not surprising, it is important to bring this aspect of the situation to the attention of the learners of mathematics. Such qualitative difference between symmetry and asymmetry can also be illustrated in the context of pizza sharing using arithmetic of fractions: whereas symmetry has only one numeric representation, $3 = 5\times{}^{3}\!/_{5}$ (Figure 2), asymmetry, as shown in Figure 3 and Figure 4, respectively, has two representations, $3 = 5\times\left({}^{1}\!/_{2} + {}^{1}\!/_{10}\right)$ and $3 = 2\times\left({}^{2}\!/_{5} + {}^{1}\!/_{5}\right) + 3\times{}^{3}\!/_{5}$. At the same time, as was mentioned in the paper, the use of fractions in teaching mathematics through pizza sharing, whatever the method, may be avoided. In the words of a future elementary school teacher, "by being able to break the problem down into pictures that students can understand, I can have more confidence in my mathematical conceptual skills which will help me explain it better …this makes math more enjoyable and fun for me, and I hope to pass this on to my future students." Whereas representation of numbers through the entities of a more complicated nature (e.g., integers through common fractions and the latter through decimals or continued fractions) is one of the big ideas of mathematics, in some cases, a big idea can first be presented through an imagery used as "a psychological tool … lifts the given function to a higher level artificial mastery of … verbal or mathematical thinking" [8]. In particular, as the last teacher's comment indicates, focusing on the images of different approaches to pizza sharing is a student-friendly educational method of connecting physical to symbolic followed by (perhaps unexpected) description of this connection through conceptual lens.

Pólya [41], the major mathematician-contributor to mathematics education in the second part of the 20th century, argued that teachers cannot impart to their students experience of mathematical discovery if they themselves have not had this experience. Such experience may be informal, without complicated mathematical machinery of formal demonstration. With this in mind, the results related to Fibonacci-like polynomials and the discovery of associated phenomena were purely computational. The computational character of the statements about real roots of the polynomials, their asymmetrical *vs.* symmetrical location, and the effect of the location on the formation of generalized Golden Ratios are in line with the methods of experimental mathematics [42,43] which, in the context of teacher education serves as a research-like experience for teacher candidates. It is worth noting that the author, being familiar with Fibonacci-like polynomials at the level of co-authoring a book on that topic [30], recognized the connection of the topic to asymmetry only by writing this paper. Indeed, "The old, the near, the accustomed, is not that *to* which but that *with* which we attend; it does not furnish the material of a problem, but of its solution" [44]. Furthermore, "looking at mathematics from the point of view of education" [6] opens a window to the duality of the abstractness of mathematics and the concreteness of its didactics using the pedagogy of computational experiments in the context of commonly available digital instruments. Eventually, such informal experiments can serve as a springboard to the formal presentation of mathematical results, highlighting asymmetry as their distinctive attribute.

## 10. Conclusion

Each of the paper's examples of the presence of asymmetry in mathematics teacher education curricula was supported by digital tools, six in all. The role of the tools in bringing asymmetry, both explicit and hidden, to the attention of teacher candidates was quite different and this difference is worthy to be mentioned in the conclusion. In the first example about pizza sharing, three of those tools—MS Word, GeoGebra, and Geometer's Sketchpad—were used just to create visually attractive images which, however, could have been drawn by hand. After all, some 50 years ago, drawing by hand was the only option available in the mathematics classroom, sometimes making it challenging (but not impossible) to distinguish between symmetry and asymmetry. The example of an inequality with parameter, borrowed from the pre-digital era advanced secondary mathematics curriculum, was supported by the Graphing Calculator and Wolfram Alpha. These tools, while making it possible to avoid the meticulous use of equivalent transformations in algebra and requisite reliance on graphing skills, could still be put aside by the present-day mathematically accomplished student like it was the case some 50 years ago. The third example about the location of roots of a quadratic trinomial relied on the use of the Graphing Calculator for an accurate construction of three regions in the plane of parameters in order to demonstrate the bifurcation phenomenon of symmetry into asymmetry and vice versa. It was the need for the accuracy of mathematical presentation that required technological support. However, the joint use of Wolfram Alpha, Maple, and the spreadsheet in the context of Fibonacci-like polynomials, stemming from asymmetrical modification of Pascal's triangle, was critical for demonstrating asymmetry and symmetry of the roots of the polynomials and the generalized Golden Ratios as strings of numbers. That is why, as was mentioned in the introduction, these findings although discussed in the context of mathematics education turned out to be new for the very discipline of mathematics. This justifies making the concluding argument about the importance of the modern-day computational tools in revealing to the learners of mathematics, including teachers of the subject matter as the major custodians of knowledge in general, the hidden presence of asymmetry inside the variety of mathematical structures.

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## Conflicts of interests

The author declares no conflict of interest.

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